

μ IOA: A Micro-Scale Foundation for Environmental-Economic Accounting

Abstract

Effective climate policy, sustainable finance, and corporate strategy require assigning environmental responsibility across products, firms, and financial flows with granular precision, system-wide consistency, and bidirectional traceability. Yet current methods are divided: Life Cycle Assessment (LCA) offers granularity but truncates system boundaries and is unidirectional, while Environmentally Extended Input-Output Analysis (EEIO) is system-consistent and bidirectional but limited by sector-level aggregation. Hybrid methods fail to entirely bridge this divide, remaining case-specific and reconciliation-dependent.

This paper introduces μ IOA (micro Input-Output Analysis), a theoretical framework that resolves this tension by constructing a fully disaggregated input-output tensor from two micro-scale data structures: the inter-entity transaction network Ω and the intra-entity process allocation model Π . Formally combining these yields a complete micro-scale IO table, on which standard Leontief and Ghosh models apply directly—producing bidirectional product and monetary-flow footprints at the most granular level.

Keywords: Micro-scale input-output analysis, Environmental footprinting, Bidirectional environmental accounting, Transaction-level data, Process allocation

1 Introduction

Effective climate policy, sustainable finance, and corporate strategy depend on assigning responsibility accurately and consistently [13, 12]. Yet a fundamental divide persists: existing frameworks must choose between the granularity of Life Cycle Assessment (LCA) and the comprehensiveness of Environmentally Extended Input–Output Analysis (EEIO). LCA provides technological detail but is constrained by truncated system boundaries [6], while EEIO captures the full economic system but remains limited by sector-level aggregation [19]. This tension undermines the reliability of firm-level disclosures, green finance instruments, and policy mechanisms like carbon border adjustments.

We observe that LCA and EEIO are, in fact, two facets of the same underlying object: the complete, micro-scale picture of the economy. To resolve the tension between them, we propose to work directly on this underlying object—formalizing the micro-scale future that leading IO scholars envisioned 13 years ago [20]: Lenzen’s “I-O-hypercubes” constructed from real-time transaction data and Suh’s industry-supplied process data yielding tables so granular that “production engineering and IOA become almost indistinguishable”.

We propose μ IOA (micro Input-Output Analysis), a methodology that enables standard IO calculations at the level of individual products, firms, and monetary flows¹. μ IOA constructs a fully disaggregated input-output system from two micro-data foundations: an inter-firm Transaction Network (Ω) mapping *product-level exchange*, and an intra-firm Process Allocation Model (Π) capturing *internal production structures*. By integrating these sources, μ IOA yields a synthesis that couples the technological granularity of LCA with the system-wide closure and bidirectionality of EEIO. Although presented here theoretically, the method is designed to align with emerging digital reporting infrastructures and data ecosystems.

To develop this framework rigorously, we first review related literature (Section 2) and recall the classical bidirectional EEIO formalism—with notation adapted to handle micro-scale price heterogeneity (Section 3). Our contribution is then threefold. First, we introduce the μ IOA formalism, which constructs a fully disaggregated input-output system from inter-firm transactions (Ω) and internal process allocation models (Π). We specifically formalize the bottom-up construction of Π from unit-process inventories, providing a “reconciliation-free” link between micro-scale engineering inventories and system-wide, macro-consistent accounting (Section 4). Second, we demonstrate that μ IOA resolves the long-standing tension between LCA and EEIO—first theoretically (Section 5), then through a numerical proof-of-concept (Section 6). Third, we examine the implementation pathways and data governance required to operationalize the framework (Section 7) before concluding (Section 8).

2 Background and literature

This section situates μ IOA within the existing body of related research, tracing methodological evolution from the two distinct pillars of environmental accounting to the various hybrid approaches that have sought to bridge them. We conclude by identifying the persistent gap that our methodology is designed to fill.

2.1 The Two Pillars of Environmental Footprinting: EEIO and LCA

Environmentally Extended Input-Output Analysis (EEIO) Framing environmental impacts as a by-product of economic activity, EEIO [1, 3] embeds environmental flows into economic models to provide *comprehensive and truncation-free* accounts of supply chain impacts. Its *additive consistency* makes it the standard for calculating national carbon footprints, analyzing policy impacts, and assessing emissions embodied in trade [13, 14]. However, EEIO suffers from the *aggregation problem* [15, 18]: reliance on sector-level averages obscures firm and product-level heterogeneity, limiting its usability for corporate and product-level decision-making.

¹We denote it μ IOA rather than μ EEIO to emphasize its generality beyond environmental applications.

Life Cycle Assessment (LCA) In contrast, Life Cycle Assessment (LCA) [47] focuses on *process-level granularity*, modeling specific technologies and multi-output processes with high precision. This makes it the method of choice for product eco-design and technological comparisons. Its primary weakness, however, is *system boundary truncation*; the necessity to arbitrarily cut off the supply chain results in incomplete and inconsistently defined footprints and an unknown, potentially significant magnitude of error [6, 19].

2.2 Bridging the Divide: Hybrid LCA-IO Methods

The tension between completeness and granularity has fostered various hybrid methods [10]. *Tiered hybrids* use EEIO emission factors to fill LCA gaps, reducing truncation but remaining case-specific and prone to double-counting [4, 28]. *IO-based* and *integrated hybrids* decompose aggregate sectors into finer sub-sectors or processes. However, these typically require *complex, iterative balancing* to reconcile disparate data scales [5, 23]. Crucially, none of these approaches offer a general, non-iterative formalism for constructing a fully disaggregated system directly from micro-scale transaction and process data—a gap μ IOA resolves.

2.3 An Orthogonal Challenge: The Duality of Analytical Direction

The analytical structure of Input-Output Analysis possesses an intrinsic duality. Alongside the demand-pull *Leontief model*—allocating impacts downstream to final consumption—the supply-push *Ghosh model* [2] allocates impacts upstream to primary inputs (value added) [11]. This paper adopts the terminology *product footprints* (downstream) and *monetary-flow footprints* (upstream) to represent these as complementary perspectives on the same flows [16].

Conventional LCA, by contrast, is inherently unidirectional and product-focused, lacking the capability to link environmental burdens directly to financial flows—such as wages and investment returns. This limits its usefulness for sustainable finance, where investment-level footprinting currently lacks a systematic accounting foundation [30].

2.4 The Emerging Frontier: Fine-Scale and Firm-Level Environmental Accounting

Sector-level intensities conceal the heterogeneity that drives errors in carbon border adjustments, green finance, and supply-chain regulations [12, 42]. This recognition has spurred parallel advances in two domains. In economics, administrative records are used to reconstruct firm-level production networks [26, 27, 36], yet these lack explicit process models and product detail. Concurrently, environmental accounting has moved toward finer granularity through high-resolution MRIO databases such as EXIOBASE [25], LCA databases such as ecoinvent [22], and Digital Product Passports [35]. However, these also either rely on aggregated structural averages or lack a unified economy-wide transaction network. While the emerging data ecosystem—e-invoicing, digital VAT, and ERP systems—is bringing micro-data within reach, a structural gap remains: the lack of a structurally-consistent formalism to integrate transaction records with process recipes in a unified IO model.

2.5 Positioning of μ IOA

As the preceding discussion has shown, despite parallel advances in fine-scale accounting, no existing framework simultaneously delivers the *consistency, granularity, and bidirectionality* required for micro-level environmental responsibility calculation. The μ IOA methodology introduced in this paper directly addresses this methodological gap.

Method	Micro-scale detail	Full-system consistency	Bidirectional imputation
EEIO	✗	✓	✓
LCA	✓	✗	✗
μ IOA	✓	✓	✓

Table 1: Core capabilities of the two pillars vs. μ IOA

3 Environmental accounting with Input-Output Analysis: Basic Formalism

This section restates the classic Leontief and Ghosh models using a unified notation designed to resolve micro-scale price heterogeneity. We use three unit types: physical (ϕ), monetary (ϵ), and environmental (ε). Capital letters denote absolute quantities (e.g., Z^ϕ), lowercase letters intensities (e.g., $a^{\phi\phi}$; intensities, as ratios, carry two units). Downstream (product) and upstream (monetary-flow) imputation are indicated by p and m superscripts, respectively. We assume a closed economy with aggregated final demand (F) and value added (V); standard orientations and algebraic conventions apply (see Appendix A).

3.1 Physical and Monetary Tables

Environmental Input-Output (EEIO) analysis allocates direct environmental impacts by propagating them through transaction networks. We define physical and monetary input-output tables, T^ϕ and T^ϵ , with associated technical and allocation coefficients (Table 5 of Appendix B.1). The standard quantity models are $X^\phi = \ell^{\phi\phi} F^\phi$ and $X^{\varepsilon T} = Vg^{\varepsilon\epsilon}$, where $\ell^{\phi\phi} = (I - a^{\phi\phi})^{-1}$ and $g^{\varepsilon\epsilon} = (I - b^{\varepsilon\epsilon})^{-1}$ are the Leontief and Ghosh inverses, respectively. Following standard EEIO practice, these are used as static *ex-post* allocation keys rather than for building predictive models.

3.2 Environmental responsibility tables and footprint calculation

The objective is to *impute* the total environmental (or social) burden (E) to the ultimate economic drivers: final-demand consumption and value-added income. Both imputation directions operate within a cradle-to-gate physical boundary: excluding use-phase and disposal. In this framework, every transaction carries dual environmental attributions.

Framing the environment as a *resource input*, impacts E are imputed downstream through the production chain to calculate **product footprints** P (intermediate transactions) and F^ε (final demand, or *Consumer Responsibility*). Conversely, framing the environment as a *pollution sink*, impacts are imputed upstream—following the monetary flow paid to suppliers—to calculate **monetary-flow footprints** M (intermediate transactions) and V^ε (value added, composed of employee compensation, profits, and taxes minus subsidies—or *Income Responsibility*). These attributions define the environmental responsibility tables (Table 6 of Appendix B.2), and are solved using the formulations in Table 2.

(p) Product embedded footprints	(m) Monetary flow embedded footprints
$x^{\varepsilon\phi p T} (I - a^{\phi\phi}) = e^{\varepsilon\phi T}$	$x^{\varepsilon\epsilon m} (I - b^{\varepsilon\epsilon}) = e^{\varepsilon\epsilon}$
$F^\varepsilon = x^{\varepsilon\phi p} \odot F^\phi$	$V^\varepsilon = x^{\varepsilon\epsilon m T} \odot V$

Table 2: The two EEIO formulations, with units adapted to μ IOA. Impacts E, here formulated as environmental intensities e, are propagated to calculate product or monetary flow intensities x, ultimately allocating to F and V.

Units and scale The unit of propagation is critical when moving from sector-scale to micro-scale. In traditional EEIO, price homogeneity makes the choice between physical and monetary units largely irrelevant. At the micro-scale, however, price heterogeneity necessitates a specific choice for consistency:

- *Downstream* (p): We use **physical units**² (ϕ). These reflect the material basis of impacts and avoid the distortions due to firm-specific pricing.
- *Upstream* (m): This direction is **inherently monetary** (ϵ). Because production inputs often involve incompatible physical units (e.g., tons of steel and liters of fuel), a physical upstream balance is impossible.

4 μ IOA: Analytical Formalism

The bidirectional EEIO framework presented in Section 3 rests on a foundational aggregation assumption: each sector produces exactly one product³. This homogeneity is more than a convenience; it underpins the formal consistency of the input-output logic. At the sectoral scale, enforcement involves reconciling Supply–Use Tables (SUTs) into a symmetric IO structure—a process that requires aggregating heterogeneous products or activities through *often-unrealistic reconciliation hypotheses* (e.g., industry-technology or product-technology assumptions). At the micro scale, however, such *forced homogeneity* is untenable: real entities produce countless outputs, share joint inputs across processes, and employ diverse technologies for identical products.

μ IOA proposes to shift from *imposing* economic structure via top-down hypotheses to *reconstructing* it directly from two micro-data foundations: digitally recorded transactions (Ω) and a synthesized process allocation model (Π). These serve as three-dimensional analogues of Supply and Use tables that combine *without reconciliation assumptions*. Figure 1 illustrates this reconstruction, formalized in three steps: 4.2 presents the micro-datasets Ω and Π - with notation 4.1; 4.3 combines them into the full input-output tensor Z_{IJKL} ; 4.4 details the construction of Π from unit-process inventories.

The resulting IO tensor generalizes the EEIO models of Section 3 to any level of disaggregation. It enables granular, consistent and bidirectional environmental accounting, free from unverifiable hypotheses or iterative balancing. The methodology is presented here at a theoretical level ; practical realization depends on data availability—a significant but achievable challenge, as discussed in 7.1.

4.1 Disaggregated IO tables

We now work at the scale of individual economic entities rather than sectors, allowing each entity to produce multiple products. The product classification is flexible: it determines footprint precision, not method feasibility (Appendix D).

Explicit axis indices Let R and S denote input and output *axes* at the {entity, product} level (see Appendix A for explicit axis notation conventions). This defines physical and monetary tables:

$$T^\phi = \begin{bmatrix} Z_{RS}^\phi & F_R^\phi \\ V_S & 0 \end{bmatrix} \quad T^\epsilon = \begin{bmatrix} Z_{RS}^\epsilon & F_R^\epsilon \\ V_S & 0 \end{bmatrix} \quad (1)$$

With \mathbb{E} entities and \mathbb{P} products, let $i, k \in \{1, \dots, |\mathbb{E}|\}$ index input and output entities, and $j, l \in \{1, \dots, |\mathbb{P}|\}$ index input and output products. We then define combined indices over {entity, product} pairs as: $r = (i-1)|\mathbb{P}| + j$ for input, or for short $r = (i, j)$; and $s = (k-1)|\mathbb{P}| + l$ for output, or $s = (k, l)$ (product index varying fastest). Thus R and S both index the set $\mathbb{E} \times \mathbb{P}$, for input and output respectively.

2D matrices Z_{RS}^ϕ and Z_{RS}^ϵ therefore represent 4D tensors Z_{IJKL}^ϕ and Z_{IJKL}^ϵ of shape $|\mathbb{E}| \times |\mathbb{P}| \times |\mathbb{E}| \times |\mathbb{P}|$,

²The method can accommodate mixed units where physical data are unavailable: see μ IOA- ϵ .

³or equivalently, a sector’s multiple output products are aggregated into a single homogeneous commodity.

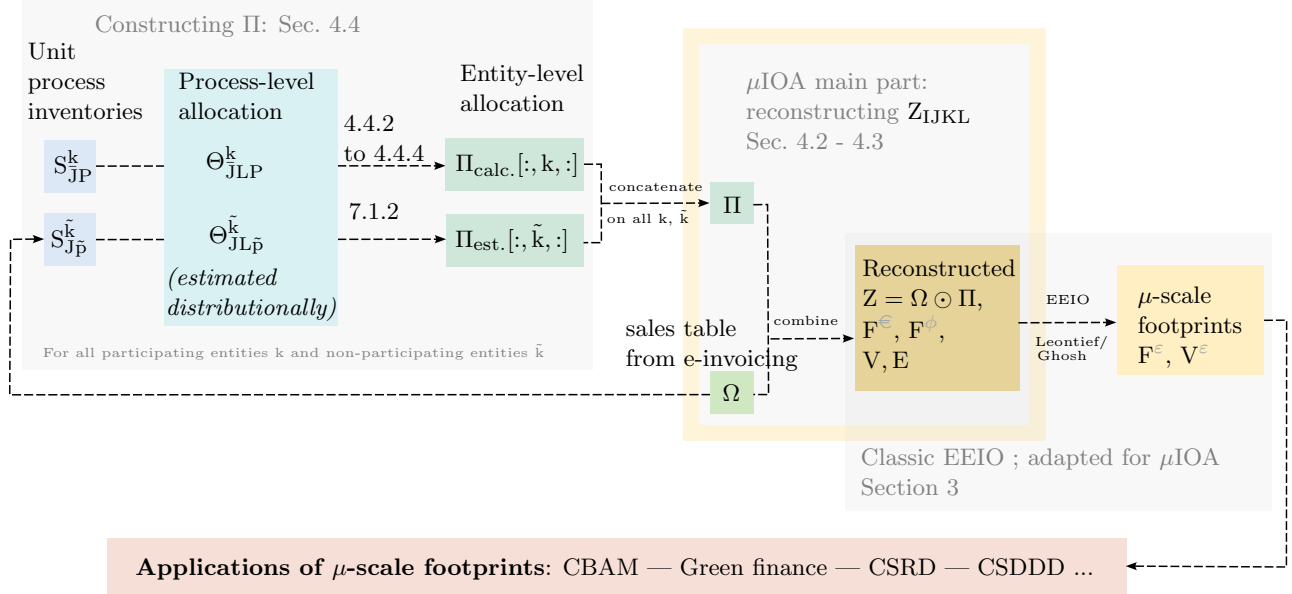


Figure 1: Overview of the μ IOA environmental footprinting methodology. The core contribution is the construction of the fully disaggregated, four-dimensional input–output tensor Z_{IJKL} (Sec. 4.2 - 4.3), which encodes the micro-scale economic structure. This tensor is derived from two micro-datasets: the Transaction Network Ω (*inter-entity product flows*) and the Process Allocation Model Π (*intra-entity transformations*). Π can be built using LCA-inspired allocation methods (4.4). Once Z_{IJKL} is obtained, standard EEIO formulas—adapted to the micro-scale—are applied (Section 3) to compute bidirectional environmental footprints at the level of individual products and monetary flows.

where (i, j, k, l) denotes:

The quantity (physical ϕ or monetary €) of input j supplied by entity i and used by entity k to produce output l .

Vectors $F_R^\phi, F_R^\text{€}, V_S$, (and E_R, E_S just after) follow the same index conventions.

Environmental footprinting Environmental footprint calculation follows Section 3, now at a finer {product, entity} granularity. As before, impacts flows support two interpretations:

- *Demand-side (consumer) responsibility*: impacts modeled as production *input*, and imputed downstream to final-demand {entity, product} pairs.
- *Supply-side (income) responsibility*: impacts modeled as production *output*, and imputed upstream to recipients of value added created in producing {entity, product} pairs.

Given fine-grained (product-allocated) direct impacts $E_S = E_R^\top$, the task is to populate the environmental tables:

$$T^{\text{€P}} = \begin{bmatrix} P_{RS} & F_R^\text{€} \\ 0 & 0 \\ E_S & 0 \end{bmatrix} \quad T^{\phi} = \begin{bmatrix} M_{RS} & 0 & E_R \\ V_S^\text{€} & 0 & 0 \end{bmatrix} \quad (2)$$

$P_{RS}, F_R^\text{€}$ (intermediate and final product footprints) and $M_{RS}, V_S^\text{€}$ (intermediate and primary monetary-flow footprints) are unknown and must be imputed from E following physical or monetary flows in T^ϕ and $T^\text{€}$.

This is identical to the classical problem of Section 3 which applies directly—only the granularity (and minor micro-specific technicalities discussed in 4.5) differ. The question is: how to obtain T^ϕ , T^ϵ , and E in practice?

4.2 Micro-Data Foundations: Transaction Records and Process Descriptions

Z_{RS}^ϕ and Z_{RS}^ϵ cannot be directly observed, but two partially aggregated versions can be obtained: the *transaction table* Ω and the *process allocation table* Π (Figure 2). These data structures directly instantiate the future envisioned in [20]: Ω gives mathematical form to Lenzen’s “IO-hypercubes”; Π realizes Suh’s vision of industry-supplied process data making “production engineering and IOA almost indistinguishable”.

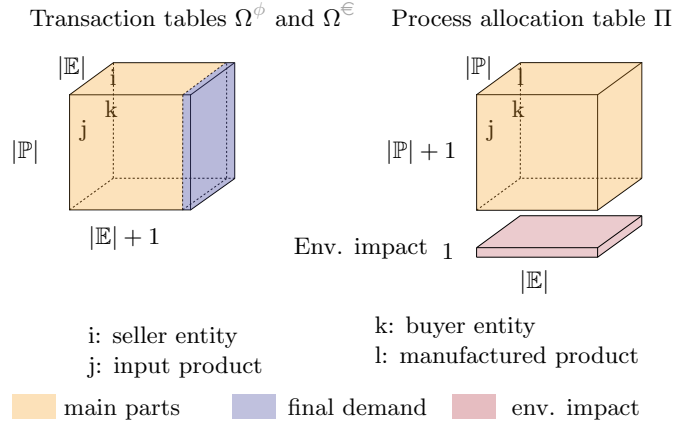


Figure 2: Input data required for fine-scale environmental footprint calculation is simply a record of economic transactions (Ω^ϕ and Ω^ϵ), and a record of internal processes of entities (Π), including allocation of direct environmental impacts.

Transaction tables Ω^ϕ and Ω^ϵ . The inter-entity transaction tables Ω_{IJK}^ϕ (physical) and Ω_{IJK}^ϵ (monetary) describe *product flows between entities*. Each element $\Omega[i, j, k]$ is the quantity (physical or monetary) of product j that entity k purchases from entity i . Both have shape $|E| \times |P| \times (|E| + 1)$, with the $+1$ in last dimension accounting for final demand. Ω^ϕ and Ω^ϵ are sparse (most entities produce few products and sell them to few buyers) and decompose as:

$$\Omega_{IJK}^\phi = \begin{bmatrix} Z_{IJK}^\phi & F_{IJ}^\phi \end{bmatrix}, \quad \Omega_{IJK}^\epsilon = \begin{bmatrix} Z_{IJK}^\epsilon & F_{IJ}^\epsilon \end{bmatrix} \quad (3)$$

Ω^ϕ and Ω^ϵ are expressed in units of product j (physical or monetary). Section 7 discusses practical sources: e-invoicing systems or business accounts.

Process allocation table Π The intra-entity process allocation table Π_{JKL} captures how each entity transforms inputs into outputs, in allocated form. Each element $\Pi[j, k, l]$ is the quantity of input product j allocated to output product l within entity k . Π has shape $(|P| + 1) \times |E| \times |P|$, where the $+1$ accommodates direct environmental use (E). It decomposes as:

$$\Pi_{JKL} = \begin{bmatrix} Z_{JKL}^\phi \\ E_{KL} \end{bmatrix} \quad (4)$$

Π is expressed in physical units of j and has no monetary equivalent, being fundamentally physical in nature⁴. Π is sparse: most entities use few inputs and produce few outputs. Section 4.4 constructs Π from *physical process recipes* and *allocation models*; Section 7.1.2 discusses practical estimation strategies.

⁴With supplier-dependent product prices, input product prices are undefined, preventing monetary conversion of Z_{JKL}^ϕ .

4.3 Constructing all required data from Π and Ω

Populating T^{P} and T^{m} requires the environmental vector $E_{\text{S}} = E_{\text{R}}^{\text{T}}$ and all components of T^{ϕ} and T^{ϵ} . We now construct these from the transaction table Ω and process allocation table Π (assumed known; sourcing discussed in Sections 4.4 and 7).

4.3.1 Reconstructing Z_{RS}^{ϕ} and Z_{RS}^{ϵ}

Marginal sums intuition The physical matrix Z_{RS}^{ϕ} corresponds to 4D tensor Z_{IJKL}^{ϕ} with $(r, s) = (i, j, k, l)$ as described in 4.1. The key insight is that observable transaction and allocation data Ω and Π are marginal sums of this tensor:

$$\Omega_{\text{IJK}}^{\phi} = Z_{\text{IJK}}^{\phi} \stackrel{\text{def}}{=} \sum_l Z_{\text{IJKL}}^{\phi} \quad \Pi_{\text{JKL}} = Z_{\text{JKL}}^{\phi} \stackrel{\text{def}}{=} \sum_i Z_{\text{IJKL}}^{\phi}$$

This provides both a consistency check between the recorded sales and allocation data (in physical units) ⁵, and a path to reconstruction, exposed now.

Source-Use Uniformity In general, a tensor cannot be reconstructed from its margins alone. Recovering Z_{IJKL}^{ϕ} from Z_{IJK}^{ϕ} (transactions) and Z_{JKL}^{ϕ} (allocations) requires one additional constraint: **for each product j and buyer k , the allocation of input j across outputs l must be independent of supplier i** . That is, the share of input j sourced from any supplier i must be the same for every output l .

This may appear as a strong constraint, but at micro-scale it follows directly from product definition: inputs that are used identically are the same product; if they were used differently they would be different products.

For each (j, k) , this **source-use uniformity** forces the 2D slice $Z_{\text{IJKL}}^{\phi}[:, j, k, :]$ (over suppliers i and outputs l) to have rank 1, thus factoring as the normalized outer product of its margins. Hence the 4D tensor can be uniquely reconstructed (5, 6).

Robustness to product classification While source-use uniformity defines the threshold for maximum footprint granularity, μIOA remains fully operational under coarser classifications. The framework then naturally calculates footprints for *composite products*—the finest granularity the data permits. See Appendix D.

Reconstruction formulas Under source-use uniformity, the 4D tensors reconstruct uniquely from their margins:

$$Z_{\text{IJKL}}^{\phi} = Z_{\text{IJK}}^{\phi} \odot Z_{\text{JK}}^{\phi (-1)} \odot Z_{\text{JKL}}^{\phi} \quad (5)$$

$$Z_{\text{IJKL}}^{\epsilon} = Z_{\text{IJK}}^{\epsilon} \odot Z_{\text{JK}}^{\phi (-1)} \odot Z_{\text{JKL}}^{\phi} \quad (6)$$

with \odot the Hadamard product, (-1) element-wise inverse, and $Z_{\text{JK}}^{\phi} = \sum_i Z_{\text{IJK}}^{\phi} = \sum_l Z_{\text{JKL}}^{\phi}$ (total physical use of product j by entity k). This normalization corrects for the total appearing in both margins, ensuring correct scale.

Two equivalent interpretations apply:

1. **Process-allocation coefficients** $\Gamma_{\text{JKL}} = Z_{\text{JK}}^{\phi (-1)} \odot Z_{\text{JKL}}^{\phi}$ allocate transaction flows across outputs:

$$Z_{\text{IJKL}}^{\phi} = Z_{\text{IJK}}^{\phi} \odot \Gamma_{\text{JKL}}, \quad Z_{\text{IJKL}}^{\epsilon} = Z_{\text{IJK}}^{\epsilon} \odot \Gamma_{\text{JKL}}.$$

⁵ $\sum_i Z_{\text{IJK}}^{\phi} = \sum_l Z_{\text{JKL}}^{\phi}$

2. **Origin coefficients** $\Lambda_{\text{IJK}} = Z_{\text{IJK}}^\phi \odot Z_{\text{JK}}^{\phi(-1)}$ spread allocations over suppliers:

$$Z_{\text{IJKL}}^\phi = \Lambda_{\text{IJK}} \odot Z_{\text{JKL}}^\phi.$$

(Applies only to physical reconstruction, as Z_{JKL}^\ominus is undefined.)

4.3.2 Side vectors: $E_S, E_R, F_R^\phi, F_R^\ominus, V_S$

Side vectors follow directly from Ω and Π via axis flattening:

$$\begin{aligned} \forall s = k|\mathbb{P}| + l, \quad E_S[s] &= E_{\text{KL}}[k, l], \quad E_R = E_S^\top \\ \forall r = i|\mathbb{P}| + j, \quad F_R^\phi[r] &= F_{\text{IJ}}^\phi[i, j], \quad F_R^\ominus[r] = F_{\text{IJ}}^\ominus[i, j] \end{aligned}$$

Value added V_S is obtained by subtracting costs from revenues for each {product, entity} combination⁶:

$$V_S = \left(\sum_s Z_{\text{RS}}^\phi + F_R^\phi \right)^\top - \sum_r Z_{\text{RS}}^\phi$$

4.4 Constructing the Process Allocation Model (Π) from Unit Process Inventories

Section 4.3 assumed Π known, but in practice it must be constructed from observable data: *physical unit process inventories*—multi-input, multi-output descriptions of elementary production steps.

For simple entities (e.g., a single-plant firm with one process), constructing $\Pi[:, k, :]$ is straightforward. For complex entities (multiple sites, integrated chains), an internal allocation procedure is required to trace every input to entity outputs.

We propose a two-step allocation model adapting LCA-inspired methods to μIOA 's firm-centered, product-disaggregated setting:

1. **Process-level allocation:** splitting multi-output process inputs among co-products—LCA's classical *allocation problem*.
2. **Entity-level allocation:** mapping entity inputs to outputs while eliminating internal intermediates.

The resulting Π feeds the 4D imputation methodology (4.1-4.3). The model below represents a theoretical ideal; Section 7.1 discusses practical sourcing pathways (entity-declared, LCA databases, hybrid) that approximate it in real-world settings.

Units: All quantities are physical (ϕ); superscript omitted for readability.

⁶ if breakdowns of employee compensation and/or taxes minus subsidies are available per product (in the form of additional entries V_{KL}^e and V_{KL}^t in Π), V_S can easily be decomposed:

$$\begin{aligned} \forall s = k|\mathbb{P}| + l, \quad V^e[s] &= V_{\text{KL}}^e[k, l], \quad V^t[s] = V_{\text{KL}}^t[k, l], \\ V^p[s] &= V[s] - V^e[s] - V^p[t], \quad V_S = \begin{pmatrix} V_S^e \\ V_S^t \\ V_S^p \end{pmatrix} \end{aligned}$$

(7)

4.4.1 Process Inventory Matrices (S_{JP}^k)

For each entity k ⁷, denote \bar{J}^k (abbreviated \bar{J}) the dimension indexing the entity’s product set $\mathbb{P}^k = \mathbb{P}_{in}^k \cup \mathbb{P}_{int}^k \cup \mathbb{P}_{out}^k$, where products are distinguished *origin and role within the entity*:

- \mathbb{P}_{in}^k : *input* products, enter from outside, **only consumed internally**;
- \mathbb{P}_{int}^k : *intermediate* products, both produced and consumed **entirely within the entity**;
- \mathbb{P}_{out}^k : *output* products, produced internally and **leave the entity**.

The same physical commodity appears as separate entries if it is produced by different internal processes, or if it serves multiple output roles (part used internally, part sold externally)

The **internal process inventory matrix** S_{JP}^k collects all unit processes operated by entity k , preserving the network of internal material flows. Rows index products $j \in \mathbb{P}^k$, columns index unit processes $p \in \mathcal{P}^k$. Entries follow LCA convention:

$$S_{JP}^k[j, p] \begin{cases} < 0 & \text{if product } j \text{ is an } \textit{input} \text{ to process } p, \\ > 0 & \text{if product } j \text{ is an } \textit{output} \text{ of process } p, \\ = 0 & \text{otherwise.} \end{cases}$$

$|S_{JP}^k[j, p]|$ is the physical flow of product j to/from process p . By construction, rows for \mathbb{P}_{in}^k are all negative, \mathbb{P}_{out}^k all positive, and \mathbb{P}_{int}^k mixed. Environmental flows and value added are treated as ordinary inputs.

S_{JP}^k thus serves as both a collection of unit-process inventories and as the signed adjacency matrix of the entity’s internal production graph.

4.4.2 Partitioning Multi-Output Processes

Multi-output processes in S_{JP}^k must be split into single-output elementary processes by allocating inputs among co-products—LCA’s classical allocation problem [7, 8].

Define a **process-level allocation matrix** Θ_{JLP}^k where $\Theta^k[j, l, p]$ is the fraction of input j allocated to output l in process p ⁸ satisfying:

$$\Theta[j, l, p] \geq 0, \quad \text{and} \quad \sum_l \Theta[j, l, p] = 1 \quad \forall j, p,$$

with $\Theta[j, l, p] = 0$ unless j is input to and l output from p . Since each process-output l corresponds uniquely to one process $p(l)$ (at this point, products are always distinguished by their producing process), the **allocated input-quantity matrix** Z_{JL}^k is:

$$Z_{JL}^k[j, l] = -S_{JP}^k[j, p(l)] \cdot \Theta^k[j, l, p(l)], \quad (8)$$

or equivalently $Z_{JLP}^k = -S_{JP}^k \cdot \Theta_{JLP}^k$.

Allocation matrices Θ may follow standard LCA rules (e.g. physical or economic). However, μ IOA’s fine-grained scale also enables data-driven alternatives: statistical or ML methods could infer $\Theta^k[:, :, p]$ from process similarities, sector benchmarks, or firm attributes, ensuring consistent allocation across similar entities without rigid LCA conventions.

⁷here k selects the entity; each has its own product set \mathbb{P}^k and process matrix S_{JP}^k , avoiding ragged tensors.

⁸Outputs l are role-specific (intermediate or final) per 4.4.1. If a physical output appears in multiple roles (l, l') , $\Theta^k[j, l, p] = \Theta^k[j, l', p]$.

4.4.3 Entity-Internal Input-Output Table

With Z_{JL}^k 's rows and columns indexing $\mathbb{P}^k = \mathbb{P}_{in}^k \cup \mathbb{P}_{int}^k \cup \mathbb{P}_{out}^k$ in this order:

$$Z_{JL}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Z_{int,int}^k & Z_{int,out}^k \\ 0 & Z_{in,int}^k & Z_{in,out}^k \end{pmatrix},$$

Inputs are never produced internally (first column block is zero); outputs are never used as inputs (first row block is zero).

Removing empty rows and columns yields the effective internal IO table:

$$Z^k = \begin{pmatrix} Z_{int,int}^k & Z_{int,out}^k \\ Z_{in,int}^k & Z_{in,out}^k \end{pmatrix}.$$

To separate flows that pass through intermediates from those already allocated to outputs, decompose:

$$Z^k = \begin{pmatrix} Z_{int,int}^k & Z_{int,out}^k \\ Z_{in,int}^k & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Z_{in,out}^k \end{pmatrix} \stackrel{\text{def}}{=} Z^{k,via} + Z^{k,dir}. \quad (9)$$

$Z^{k,via}$ describes flows that pass through internal intermediates; $Z^{k,dir}$ captures inputs already allocated directly to final outputs.

4.4.4 Calculating Entity-level Input-output Allocation

In decomposition (9), $Z^{k,dir}$ already maps entity inputs to entity outputs via its lower-right block $Z_{in,out}^k$. To obtain the entity-level allocation $\Pi_{JKL}[:,k,:]$, we must also map indirect flows: allocate inputs passing through $Z^{k,via}$ to outputs while eliminating internal intermediates.

Treat $Z^{k,via}$ as a small input-output system in physical units ϕ :

$$Z^{k,via} = \begin{bmatrix} Z_{int,int}^k & Z_{int,out}^k \\ Z_{in,int}^k & 0 \end{bmatrix}, \quad X^{k,via} = [Z_{int,int}^k \quad Z_{int,out}^k] \mathbf{1}.$$

Define the input-intensity matrix

$$\mathbf{a}^{k,via} = \begin{bmatrix} Z_{int,int}^k \\ Z_{in,int}^k \end{bmatrix} \widehat{X^{k,via}}^{-1} \stackrel{\text{def}}{=} \begin{bmatrix} a_{int,int} \\ a_{in,int} \end{bmatrix}.$$

Allocating inputs to outputs via intermediates follows the Leontief product footprint model (Table 2), but (i) propagating *multiple input intensities* (rows of $\mathbf{a}_{in,int}$) rather than a single environmental intensity $e^{\varepsilon\phi}$; (ii) propagating them over multiple output columns (multi-output block $Z_{int,out}^k$) instead of a single ‘final-demand’ column (\mathbf{F}^ϕ); (iii) aggregating over *both origin and destination* intermediate dimensions.

Solve for intensities \mathbf{x}_{int} ⁹:

$$\mathbf{x}_{int}^\top [i, :] (\mathbf{I} - \mathbf{a}^{k,via}) = \mathbf{a}_{in,int} [i, :] \quad (10)$$

Then obtain indirect allocation:

⁹compared to Table 2 left, $\mathbf{a}_{in,int} [i, :]$ plays the role of $e^{\varepsilon\phi^\top}$ and $Z_{int,out}^k [i, j]$ plays the role of \mathbf{F}^ϕ

$$Z_{\text{in,out}}^{\text{k,via}}[i,j] = \mathbf{1}^\top (\mathbf{x}_{\text{int}}[:,i] \odot Z_{\text{int,out}}^{\text{k}}[:,j]) \quad (11)$$

Equation (10) collapses the input ‘int’ dimension; (11) collapses the output ‘int’ dimension, yielding a clean input-to-output mapping. Computing for all (i,j) gives the full indirect allocation $Z_{\text{in,out}}^{\text{k,via}}$.

Total entity-level allocation is then:

$$Z_{\text{in,out}}^{\text{k,tot}} = Z_{\text{in,out}}^{\text{k,via}} + Z_{\text{in,out}}^{\text{k}} \quad (12)$$

where $Z_{\text{in,out}}^{\text{k}}$ is the direct, non-zero block from $Z^{\text{k,dir}}$.

After embedding entity inputs and output products into global product classification \mathbb{P} , this matrix becomes $\Pi_{\text{JKL}}[:,k,:]$ —rows index global inputs J, columns global outputs L, and entries are physical quantities of input j allocated to output l within entity k. This completes Π ’s construction from unit process inventories.

4.5 System Resolution Subtleties in μIOA

With the μIOA components of Equation 1 and direct impacts E established, footprint intensities \mathbf{x} are obtained by solving the linear systems defined in Table 2 on the compressed subspace of entity-product combinations where total output $X > 0$ —i.e., after removing rows and columns corresponding to non-existent production—ensuring the coefficient matrices a and b are well-defined.

$$\mathbf{x}^{\varepsilon\phi\text{p}^\top} (\mathbf{I} - \mathbf{a}^{\phi\phi}) = \mathbf{e}^{\varepsilon\phi^\top} \quad \text{and} \quad \mathbf{x}^{\varepsilon\in\text{m}} (\mathbf{I} - \mathbf{b}^{\in\in}) = \mathbf{e}^{\varepsilon\in} \quad (13)$$

where \mathbf{e} represents the direct impact intensities per unit of output ($\mathbf{e}^{\varepsilon\phi^\top} = \mathbf{E}^\top \odot \mathbf{X}^{\phi^{-1}\top}$) or revenue ($\mathbf{e}^{\varepsilon\in} = \mathbf{E} \odot \mathbf{X}^{\in^{-1}\top}$). In contrast to conventional sectoral IOA, where Leontief and Ghosh inverses are typically well-defined, resolving the *micro-scale* system involves structural nuances arising from the high resolution of the underlying data.

4.5.1 Product Coupling and System Constraints

In μIOA , global linear systems in Table 2 (and entity-internal systems (10)) are structurally over-determined. This arises from two forms of coupling: (1) *process coupling*, where sequential transformations within an entity create linear dependencies between consecutive process scales (and direct impacts), and (2) *co-product coupling*, where fixed-ratio outputs share a single production recipe Π .

Consequently, the matrices $(\mathbf{I} - \mathbf{a}^{\phi\phi})$ and $(\mathbf{I} - \mathbf{b}^{\in\in})$ are singular. A solution \mathbf{x} exists only if the direct impact vector \mathbf{e} is consistent with these internal constraints—i.e., \mathbf{e} must lie within the column space of the system matrix. While this requires ensuring data consistency before resolution, it is not a mathematical flaw; rather, it reflects the model’s ability to enforce physical and structural conservation laws across the disaggregated economy.

4.5.2 Non-Positive Value Added and Product Grouping

While the inter-entity transaction network Ω and intra-entity allocation Π represent physical causality, their interaction with market prices can result in non-positive value added components for specific entity-product pairs in the disaggregated monetary table \mathbf{T}^{\in} . This typically occurs with unavoidable byproducts (e.g., industrial scrap or waste heat) where physical allocation (θ) assigns significant input footprints to an output with small or negative market value.

In such cases, the upstream (Ghosh) system becomes ill-conditioned. Structurally, this indicates that these products are not independent drivers of revenue but are ‘subsidized’ by the entity’s primary outputs. To

ensure a stable solution when propagating monetary-flow footprints, we define a grouping matrix G that aggregates the products of each entity into profitable clusters¹⁰. We then solve:

$$\mathbf{x}^{\text{€m}} G(\mathbf{I} - \mathbf{b}^{\text{€€}}) = \mathbf{e}^{\text{€€}} \quad (14)$$

The resulting upstream footprints $\mathbf{x}^{\text{€m}}$ are defined per entity-product group. This grouping is required only for the upstream *Income Responsibility* model; the downstream *Consumer Responsibility* model remains fully disaggregated, as physical footprints are independent of sale prices.

5 μ IOA: Unifying LCA and EEIO

With this formalism now in place, this section reinterprets established environmental accounting methods as specific, partial representations of the underlying μ IOA tensor. By framing EEIO, LCA, and hybrids through this lens, we reveal the specific data subsets and consistency compromises each method employs (summarized in Table 3)—and crucially, derive testable predictions about their error patterns. Section 6 will confront these predictions numerically.

5.1 EEIO as Macro-scale Aggregation

EEIO represents a macro-scale projection of the μ IOA tensor. Formally, it is obtained by (1) *Aggregating over product axes* (J and L), which treats all products within a sector as homogeneous [10, 19]. (2) *Grouping entities* (I and K) into sector dimensions (\bar{I} and \bar{K}) via summation over membership indices.

This results in the standard sector-by-sector table $Z_{\bar{I}\bar{K}}$. While this ensures global consistency, the aggregation obscures differences between individual firms and their specific internal production processes [42]. In short, EEIO provides a comprehensive economy-wide view but sacrifices micro-scale granularity and ignores the price and product heterogeneity captured in μ IOA.

5.2 Firm-level Network Models

Firm-level models (e.g., from VAT data) represent the μ IOA tensor aggregated over all product axes (J, L). This entity-to-entity matrix Z_{IK} preserves the granular topology of the production network but loses all product-specific and process-level detail [17, 26]. While these models effectively trace monetary flows between firms, they are ‘blind’ to the specific nature of the goods traded or the internal transformations within each entity. Consequently, they cannot distinguish between different technological pathways or multi-output allocation logic.

5.3 Life Cycle Assessment (LCA)

LCA operates on *unit-process inventories* which, in the μ IOA framework, correspond to columns of internal process inventory matrices S^k (4.4.1). By tracing specific technological pathways through these inventories, LCA achieves high physical granularity and handles multi-output allocation at the process level [8].

Crucially, however, these LCA pathways are *structurally truncated*: Upstream ends are typically assigned exogenous background data, while downstream ends are treated in isolation from the broader economy, not feeding back to any inputs. While μ IOA retains the full process detail of LCA through the internal logic of S and Θ , it does not truncate any ends; instead, it embeds all processes within a closed transaction structure Ω to ensure economy-wide consistency [6].

¹⁰Because cost structures are entity-specific, G must be defined over the full disaggregated dimension $R = I \times J$.

Method	Process Granularity	Transaction Granularity	Boundary	Key Limitation
EEIO	Sector average	Sector average	Full	Aggregation bias
Firm-level Networks	None	Entity-level	Full	No process detail
LCA	Unit-process	Modeled pathway	Truncated	Truncation error
Tiered Hybrid	Unit-process + sector avg.	Modeled pathway	Partial	Aggregation + truncation
IO/Integrated Hybrid	Mixed	Mixed	Full	Reconciliation problem
μIOA	Unit-process (via Π)	Entity-product (via Ω)	Full	Data requirements

Table 3: Comparison of environmental accounting methods in relation to the μ IOA data structure.

5.4 Hybrid Methods

Hybrid methods attempt to bridge the gap between LCA and EEIO but differ in how they manage the interface between scales. In *Tiered Hybrid Analysis*, the relationship is unidirectional: sectoral emission factors from EEIO are used to ‘plug’ the upstream ends of an LCA pathway, but the LCA data do not influence the IO model [10]. Consequently, Tiered Hybrid simply trades truncation error for the aggregation bias of the background IO data [6, 19].

In contrast, *IO-based Hybrid* (Disaggregated Hybrid) and *Integrated Hybrid Analysis* attempt a structural merger by embedding LCA-derived processes directly into the IO matrix. This yields a mixed-granularity system where a foreground (LCA) is solved alongside a background (EEIO) as a single system:

$$\mathbf{a}^{\text{hyb}} = \begin{bmatrix} \mathbf{a}_{\text{ff}} & \mathbf{a}_{\text{fb}} \\ \mathbf{a}_{\text{bf}} & \mathbf{a}_{\text{bb}} \end{bmatrix} \quad (15)$$

Despite their pragmatic utility, these methods are essentially *reconciliation-driven*: because the foreground and background originate from different observational scales, they represent inconsistent constraints on the system flows. Through the lens of μ IOA, this mismatch is a direct result of using inconsistent projections of Z_{IJKL} : the background \mathbf{a}_{bb} is a macro-aggregation (Z_{IK}), while the foreground \mathbf{a}_{ff} follows (j, k, l) pathways for ‘representative’ producers $\{\bar{k}\}$. Restoring balance typically requires iterative proportional fitting (e.g., scaling background entries while keeping process data fixed), which inevitably distorts either the process detail or the macroeconomic consistency [9].

In contrast, μ IOA is *construction-driven*: it analytically builds the unified tensor from observed transactions (Ω) and process inventories (S^k), bypassing the reconciliation problem entirely.

5.5 Summary Comparison

Table 3 compares established environmental accounting methods with the proposed μ IOA methodology through the lens of Z_{IJKL} :

The table crystallizes a persistent tension: EEIO sacrifices detail for coverage; LCA and tiered hybrid preserve process detail but truncate the transaction network; and integrated hybrids attempt to merge both but introduce a reconciliation problem. μ IOA resolves this tension by constructing Z_{IJKL} directly from Ω and Π , delivering bidirectionality, granularity and closure without reconciliation— at the cost of demanding micro-data, a requirement becoming plausible with e-invoicing and process-level reporting.

6 μ IOA Proof-of-Concept: A Micro-Scale Numerical Illustration

While a full-scale implementation of μ IOA would require micro-data not yet systematically available, its logic, computational feasibility, and numerical benefits can be demonstrated on a minimal, controlled example. This section presents such a proof-of-concept on an artificially generated, intentionally minimal toy economy that

- *Circular material flow.* Scrap metal generated by both manufacturers is purchased by `SteelCo_EF` as an input to electric arc steelmaking. This closed-loop recycling introduces a feedback in the physical supply chain that truncated methods cannot capture.
- *Distinct final consumption patterns.* Final consumers purchase electricity, steel, cars, and services, with preferences distributed across available suppliers.

The complete process recipes, sourcing preferences, and constraint parameters are provided in Appendix F.1.

6.2 Generating a balanced micro-economy

This toy economy is parameterized by process scales, sourcing fractions, transaction prices, and per-process labor costs. Initial parameters do not automatically satisfy physical market-clearing conditions nor economic viability constraints expected of a realistic economy. To obtain a consistent micro-economic dataset, we solve a constrained optimization problem.

All free parameters are optimized simultaneously via gradient descent to achieve: exact material balance for non-final-demand products; final demand quantities within target ranges; sourcing fractions close to prescribed preferences; and entity markups and prices within plausible ranges (including green premiums where applicable).

The optimization converges to a fully balanced state satisfying all constraints. Appendix F.3 provides further details on this balancing procedure; the resulting dataset (F.2) is the basis for all subsequent footprint calculations.

6.3 Footprinting methods benchmarked

We compare five methods applied to the same balanced dataset; only the information available to each method varies.

μ IOA-full (baseline). The full methodology of Sections 3-4, with complete access to both transaction network Ω and internal production recipes S^k for all entities. Process-level allocation matrices Θ (`process_recipes`, F.1) differ across the two `Stamping` processes to illustrate that allocation rules can vary by entity. These yield entity-level allocation Π via 4.4. All entity-product profits are positive, so no VA-side grouping (4.5.2) is required for upstream footprints.

μ IOA-partial. Simulates incomplete data: `ManuCo_Basic` does not disclose its internal process structure S^k . Its allocation Π is estimated by applying *entity-level* allocation rules Θ of a similar participating entity (`ManuCo_Advanced`), as outlined in 7.1.2.

Process-based LCA. A process-level LCA performed separately on each car type (`C` from `ManuCo_Advanced` and `ManuCo_Basic`), following typical practitioner practice. The system boundary is cut off after two upstream tiers (`depth=2`): inputs at the cutoff depth are assigned *direct* emission factors—mimicking, within this shallow economy, the truncation of a real study where tracing stops before reaching more distant upstream supplies. Scrap inputs are truncated entirely (assigned zero), reflecting the common difficulty of tracing recycled material flows. The internal loop at `ManuCo_Basic` is included—a practitioner would notice and correct for such an obvious feedback. No supplier averaging is applied—supply chains are followed individually, which is already more detailed than most process LCAs.

μ IOA-€. Identical to μ IOA-full but uses monetary units also for downstream (product) footprint imputation. This isolates the contribution of price heterogeneity to the total error of sector-level EEIO. (Upstream imputation is unchanged, as it is inherently monetary in both cases.)

Sector-level EEIO. Firms are aggregated into four sectors (Energy, Materials, Manufacturing, Services) as defined in F.1. A symmetric input-output table is constructed by aggregating monetary transactions to the sector level, and standard bidirectional EEIO (Leontief downstream, Ghosh upstream) is applied.

Method	Effect	Downstr. (F) APE (%) <i>(min - weighted mean - max)</i>	Upstr. (V) APE (%) <i>(min - weighted mean - max)</i>
μIOA			
μ IOA-full	Reference	0-0-0	0-0-0
μ IOA-partial	Some process data missing	0-2.8-19	0-0.03-34
Process-based (<i>product-level, truncated</i>)			
LCA (cars only)	Truncated system	0.7-6.9-23	n/a
EEIO-style (<i>methodological simplifications</i>)			
μ IOA- ϵ	Monetary units downstream	0.1-1.4-2.9	0-0-0
EEIO	Product and entity aggr.	20-89-610	33-100-2700

Table 4: Numerical evaluation of footprinting methods, compared to μ IOA-full.

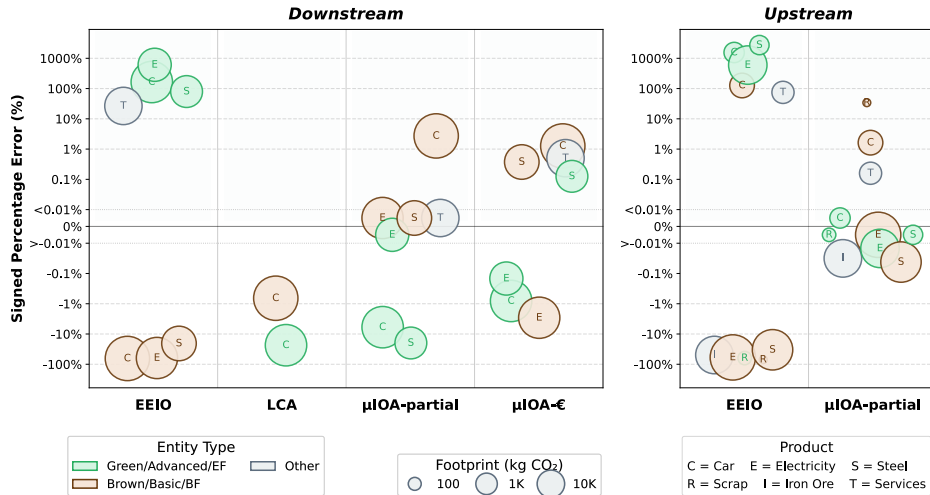


Figure 4: Percentage Error by method and direction, per {entity, product}.

6.4 Evaluation metrics

For each method, we compute downstream footprints F^ϵ (of final demand products) and upstream footprints V^ϵ (of total value added, i.e., labor plus profits).

For every {entity, product} pair with non-zero true footprint, we calculate the Absolute Percentage Error (APE) relative to μ IOA-full, which serves as the baseline because unlike other methods it has access to all micro-data (Ω , S^k , Θ) without aggregation, truncation, or estimation—representing the true micro-economic state. We report the minimum, weighted mean (weighted by true footprint magnitude), and maximum APE across all pairs. The weighted mean ensures the summary is dominated by ecologically significant flows.

6.5 Results and discussion

Table 4 reports these summary statistics for each method and direction; Figure 4 displays the full distribution of errors for every {entity, product} pair.

μ IOA-partial Withholding `ManuCo_Basic`'s internal process structure (its Π estimated from `ManuCo_Advanced`) over-allocates impacts to cars (0.99 vs Basic's true 0.96). Downstream, this inflates basic car footprints (+2.7%), but also deflates basic scrap footprint, which reduces `SteelCo_EF`'s steel footprint (-19.8%) and in turn `ManuCo_Advanced`'s car footprint (-5.9%). Upstream, reallocating inputs toward cars shifts costs away from scrap without changing revenue. Scrap's value added therefore increases, attracting more upstream

footprint (+34%). Cars bear very slightly higher costs, marginally reducing their value added, but loop effects through the system still yield a small net increase (+1.6%).

Process-based LCA Truncating the scrap loop produces systematic underestimation for the advanced car (-23.5%), traceable to **SteelCo_EF**'s underestimated footprint (LCA: 872 kgCO₂/ton; μ IOA: 1210 kgCO₂/ton). The basic car error is low (+0.66%) because its steel comes from **SteelCo_BF** (which uses iron ore, not scrap) and the internal loop is correctly included. This isolates truncation as the sole source of error for the advanced car, as anticipated in 5.3.

μ IOA-€ For identical products sold to different buyers, prices vary by ± 1 –3% in the balanced economy (e.g., **SteelCo_EF** steel: 912–922 EUR/ton; **PowerCo_Green** electricity: 83–87 EUR/MWh). These variations—realistic and even conservative compared to potential real-world price differences—shift footprint allocation relative to using physical flows, introducing small but systematic errors (weighted mean 1.4%). This error is negligible compared to EEIO's aggregation error (90%), confirming that while physical units are preferable in μ IOA when available, price heterogeneity is not the primary source of EEIO's inaccuracy.

Sector-level EEIO Aggregating firms into four sectors produces the largest errors: downstream weighted mean 90% (range 21–610%), upstream 101% (range 33–2736%). Green producers are most distorted because their lower volumes are swamped by brown counterparts in sector averages: **PowerCo_Green**'s downstream footprint is over-allocated by 610%, **SteelCo_EF**'s upstream by 2736%. These aggregation errors propagate throughout the system: even sectors with a single firm (**ServiceCo**, **MiningCo**) show 26–75% errors because they purchase from upstream sectors whose heterogeneity is masked by aggregation. The magnitude of aggregation bias observed here is broadly consistent with simulation results by [40, 46].

Synthesis The numerical results crystallize the theoretical claims of Section 5. For LCA, the advanced car's -23.5% error from scrap truncation is only one facet of a deeper structural incompleteness. Process-based LCA is labor-intensive and static—typically applied to a handful of products and rarely updated—and fundamentally unidirectional, providing product footprints but no upstream income-based counterpart. Sector-level EEIO, by contrast, is comprehensive and bidirectional, but aggregation bias is severe: weighted mean errors exceed 90%, with much larger errors for small producers. μ IOA avoids both traps. It handles incomplete process data or missing physical quantities gracefully (errors localize, not corrupt the system), and delivers consistent, automatable footprints in both directions.

7 Operationalizing μ IOA: Data, Governance, and Implementation.

The μ IOA formalism provides a complete theoretical solution to the granularity–completeness tension in environmental-economic accounting. Translating this formal tensor structure into a functioning system presents a distinct class of challenges—data, computation, and governance—that define a concrete research and implementation agenda at the intersection of input-output economics, industrial ecology, and data science. This section outlines that agenda, arguing that while the required data ecosystems are emerging, the main barriers to large-scale adoption are institutional rather than technical.

7.1 Sourcing the Core Data: A Pragmatic Roadmap

A practical implementation of μ IOA requires access to two foundational datasets: the transaction table Ω and the process table Π . Currently, neither is systematically available at the required micro-scale. However, the necessary data points already exist within corporate and fiscal digital silos. This subsection outlines a pragmatic pathway to assemble these digital traces by building on emerging infrastructures, extending existing accounting practices, and adopting a phased, value-driven approach to data collection.

7.1.1 Transaction Data (Ω): The Institutional Pathway

The transaction network Ω (inter-firm product flows) is *already recorded* in corporate accounting systems: the barrier is not creating the data, but establishing standardized, privacy-preserving mechanisms to leverage it for environmental footprinting.

Crucially, this is becoming institutionally feasible through mandatory e-invoicing regimes—such as Italy’s *SdI*, Brazil’s *NF-e*, and the forthcoming EU *VAT in the Digital Age* (ViDA) directive—which capture the firm–product granularity required for Ω [33, 44]. When combined with secure data-pooling arrangements—such as trusted data trusts or ‘computation-under-secrecy’ models—this infrastructure can provide Ω at scale without exposing sensitive commercial details. The remaining obstacle is thus governance: defining legal and operational protocols for accessing and using this data for authorised environmental-economic analysis.

7.1.2 Process Data (Π): From Unit Inventories to Allocation

The process table Π captures how each firm transforms inputs into outputs. Ideally, Π is constructed from two elements: (1) *Unit Process Inventories* (S), recording physical flows for internal production steps; and (2) *Allocation Matrices* (Θ), which define how inputs are split among co-products (4.4).

To ensure transparency and prevent strategic reporting, we propose a *separation of concerns*: firms report verifiable physical inventories (S), while the system applies consistent, allocation rules (Θ) centrally (rule-based or *data-driven*—leveraging sectoral benchmarks, engineering priors, or statistical patterns across similar participating firms to ensure the allocation logic remains objective and comparable.) This separation establishes a transparent and auditable foundation for process data.

Where proprietary firm-level data S is unavailable, μ IOA *degrades gracefully*. A non-participating entity \tilde{k} can be modeled as operating a single-process \tilde{p} described by $S_{J\tilde{p}}^{\tilde{k}}$, where J indexes the entity’s externally traded inputs and outputs—all of which are observable from the transaction table Ω (see Figure 1). Determining $\Pi[:, \tilde{k}, :]$ for this entity then reduces to estimating a single $\Theta_{JL\tilde{p}}$ from firm inputs to firm outputs.

In this mode, μ IOA still eliminates the sector and product-averaging bias of EEIO by retaining firm-specific transaction granularity.

7.1.3 A Phased Implementation Path

Implementing μ IOA does not require a monolithic upfront investment. The framework supports a modular rollout—growing in accuracy and delivering value at each stage of data availability—thereby realizing the future IO scholars envisioned [20]: merging international governance mechanisms with individual-driven supply-chain literacy compelling industry disclosure.

- *Phase 1: Automated Prototypes.* Immediate micro-scale footprints are generated by pairing observed transaction data (Ω) with *approximated* process structures (Π).
- *Phase 2: Targeted Refinement.* The system identifies high-impact “hotspots,” triggering the collection of ground-truth inventories (S^k) from key emitters or industry leaders to replace approximations.
- *Phase 3: Institutionalization.* Reporting requirements are aligned with emerging regulations (e.g., CSRD, Digital Product Passports), creating a self-sustaining cycle of high-resolution data.

A detailed expansion of this example roadmap is provided in Appendix E.

7.2 System Engineering: From Formalism to Operation

Translating the μ IOA formalism into a global operational system presents four primary engineering challenges. These represent a focused technical agenda for transitioning the theory into a functioning digital infrastructure.

Accrual-based Accounting (Stocks and Capital). To align with standard financial reporting, μ IOA must relax static balance equations to allow for *environmental amortization*. Distributing the impacts of capital goods, durable intermediates, and material stocks over their actual periods of economic use prevents the temporal distortion of footprints and enables consistent, high-frequency tracking.

Uncertainty Propagation and Precise Quantification. Because Ω , S , and Θ inevitably contain measurement errors or statistical estimates, the framework must systematically propagate these uncertainties through the linear system. This allows for a precise quantification of the uncertainty in final results and a decomposition of footprints by data quality.

Privacy-Preserving Computation. Commercially sensitive transaction and process data require technical safeguards, such as secure multi-party computation or trusted execution environments. These architectures allow for the calculation of verified aggregate footprints without exposing raw, firm-specific data to any single party.

Scalable Computation and Solvability. While a global network involves billions of nodes, the *sparsity* and *hierarchical structure* of production networks make this a tractable numerical task. Exploiting this structure allows for the design of problem-specific solvers that can handle the massive scale of the 4D tensor.

Addressing these fronts ensures that the μ IOA framework is not only theoretically sound but also a robust and auditable foundation for environmental intelligence.

7.3 Governance, Incentives, and Broader Applications

The technical challenges of μ IOA are tractable; the deeper bottleneck is institutional. Realizing this analytical architecture requires building governance structures and aligning incentives across a multi-stakeholder ecosystem.

Short-term Call to Action. A local μ IOA *Prototype* is feasible today, following an emerging global trend of repurposing administrative micro-data for supply-chain mapping—notably via Brazil’s *Nota Fiscal Eletrônica* (NF-e) system [45] and recent enterprise-level global databases[43]. Provided access to Ω is granted under strict tax-secrecy protocols, a prototype will demonstrate feasibility and build trust. Firms should simultaneously begin reporting unit process data (S^k) in standardized formats to support next phases.

Governance and Standardization. International standards for Ω and S^k (via OECD, UN, or ISO) are essential to prevent fragmentation. Public-private data trusts can pool sensitive data securely, enabling the benchmarking and validation required for global scaling. A phased rollout, starting with pilot supply chains or regulated sectors, ensures practical learning and builds credibility.

Fiscal and Regulatory Drivers. Public procurement, representing up to 15% of GDP in advanced economies, provides a powerful lever to mandate participation by linking contracts to unified accounting. This lever complements a shifting regulatory landscape: frameworks such as CSDDD and CBAM require precise product footprints, while sustainable finance increasingly necessitates tracking impacts tied to value-added recipients (*income responsibility*). For consistent and equitable reporting, the CSRD must eventually integrate both perspectives. Ultimately, these regulations create a strong and immediate demand for the μ IOA architecture, as to our knowledge, it provides the only mathematically consistent pathway to reconcile granular, source-defined firm declarations with global supply-chain footprints.

Individual-level Accountability and Fairness. By mapping flows to individuals, μ IOA can underpin progressive environmental taxation and personalized carbon budgets[21, 37]—tools currently hindered by data gaps. Centrally performed allocation via transparent rules (Θ^P) acts as a safeguard against strategic ‘greenwashing’ or gameable reporting by firms.

μ IOA as Economic Infrastructure. Beyond environmental accounting, this granular data serves as foundational infrastructure for climate-risk assessment[29], inflation tracking[31], and supply-chain resilience mapping[27, 36]. These co-benefits broaden the coalition of supporters—including central banks, competition authorities, and industrial planners—justifying the investment in the underlying data ecosystem.

8 Conclusions

Environmental footprinting has long been divided between the granular but truncated and unidirectional perspective of Life Cycle Assessment (LCA) and the comprehensive and bidirectional but aggregated view of Environmentally Extended Input–Output analysis (EEIO). This paper has proposed μ IOA, a micro-scale input–output formalism that resolves this tension by constructing a fully disaggregated economic representation from two emerging data structures: the Transaction Network (Ω , “who buys what from whom”) and the Process Allocation Model (Π , “how firms transform inputs into outputs”).

The paper has made three contributions, as set out in the introduction: (1) introducing the μ IOA methodology, detailing construction from transaction records and process data; (2) showing—conceptually and through a numerical proof-of-concept—that μ IOA merges LCA’s granularity with EEIO’s completeness and bidirectionality, eliminating aggregation bias and truncation error; and (3) examining implementation pathways and data governance required to operationalize the framework.

If realized, the architecture carries powerful implications: for policy—a consistent micro-scale basis for carbon border adjustments, green finance, and supply-chain regulations; for firms and investors—transparent, auditable footprinting at product and payment levels, with preserved commercial confidentiality; for research—a micro-economic infrastructure bringing fine-grained data into economic models.

As of today, the framework remains theoretical, but its potential benefits justify tackling its associated challenges. Trusted data-sharing mechanisms, privacy safeguards, and international standards must balance transparency with confidentiality. Concurrently, scalable tensor methods—including large scale solvers, handling of capital stocks, and propagation of uncertainty must be developed to deliver actionable outputs at scale. Practical pilots can begin immediately—validating the approach, enabling granular applications (e.g., economic shock propagation, climate-risk assessment), and iteratively improving data governance.

In sum, μ IOA moves environmental–economic accounting from reconciliation-driven compromises to a construction-driven, consistent micro-foundation. By working directly on the observable microstructure of the economy, it offers a coherent and scalable vision for transparent, equitable, and effective sustainability governance in the digital age.

Data statement

The code for reproducing the numerical proof-of-concept is available at <https://fileditchfiles.me/file.php?f=/b79/sPnkelIqnYuATjeslh1F.zip> and will be made publicly available upon publication.

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Supplementary Materials

A Notation Summary

This appendix details the mathematical conventions and provides a comprehensive reference for the notation used throughout the paper.

A.1 Mathematical Conventions

Algebraic notation We use standard matrix and tensor notation. For a vector \mathbf{a} , $\hat{\mathbf{a}} = \text{diag}(\mathbf{a})$ denotes the square matrix with \mathbf{a} on its diagonal. The element-wise (Hadamard) product is denoted $\mathbf{a} \odot \mathbf{b}$; for matrices and tensors, broadcasting over missing indices is implicit. Column scaling of a matrix \mathbf{A} by (row) vector \mathbf{b}^\top is then written $\hat{\mathbf{A}}\mathbf{b} = \mathbf{A} \odot \mathbf{b}^\top$, row scaling as $\hat{\mathbf{b}}\mathbf{A} = \mathbf{A} \odot \mathbf{b}$. Element-wise inversion is denoted $\mathbf{a}^{(-1)}$. Matrix multiplication satisfies $\mathbf{A}\mathbf{b} = (\mathbf{A} \odot \mathbf{b}^\top)\mathbf{1}$ or $\mathbf{b}^\top\mathbf{A} = \mathbf{1}^\top(\mathbf{b} \odot \mathbf{A})$, where $\mathbf{1}$ is a column vector of ones. We use classical notation ($\hat{\cdot}$, ordinary matrix multiplication) where it preserves familiarity with standard input-output conventions, and element-wise notation (\odot , $^{(-1)}$) when seeking maximum clarity in scaling, normalization, or tensor operations.

Explicit axis notation In Section 4 and onwards, fine-grained analysis employs explicit index notation (e.g., Z_{IJKL}) with capital letters denoting dimensions (I=input entity, J=input product, etc.). This constitutes a simplified notation of Einstein's summation notation in fully Euclidean context where indices serve only as axis labels and can all be written in the same position (i.e. no need to raising or lowering because the underlying metric is the identity). Implicit summation over repeated indices (Einstein convention) is used where appropriate.

Absolute value vs intensities Over the whole paper, we manipulate absolute-value quantities (e.g. physical quantity associated to a transaction) and intensities (e.g. input intensities). Regardless of dimension, absolute-value quantities are denoted with capital letters (e.g. Z^ϕ) and intensity quantities with lowercase letters (e.g. $a^{\phi\phi}$). In general, absolute-value and intensity forms of the same quantity use the same letter (e.g., $F^\phi \rightarrow f^{\phi\phi}$). The only exceptions are the square, central intensity coefficients: for example the Leontief coefficient $a^{\phi\phi}$ and the Ghosh coefficient $b^{\phi\phi}$, both derived from Z^ϕ . For these we keep the standard notation \mathbf{a} and \mathbf{b} to remain consistent with the IO literature.

Units We manipulate three types of units: physical units (ϕ), monetary units (ϵ), and environmental units (ε). Unlike orientations which are implicit, we denote units explicitly with superscripts. For example, Z^ϕ is expressed in physical units. Intensities, as ratios, have two superscripts: for example $a^{\varepsilon\phi}$ is expressed in environmental unit per physical unit.

Dimensions and scope For this paper, and without loss of generality, we place ourselves in the simplified context of an isolated economy without imports or exports, and aggregated final demand and value added. Dimensions and orientations are implicit and follow the usual convention in input-output analysis: for example value added \mathbf{V} is always a row vector, intermediate transactions Z^ϕ is a square matrix, and final demand \mathbf{F}^ϕ is always a column vector. Quantities used in both transpositions (e.g. \mathbf{E}) are written \mathbf{E} and \mathbf{E}^\top .

	Physical units (ϕ)	Monetary units (ϵ)
(A) abs. values	$\begin{bmatrix} Z^\phi & F^\phi \\ V & 0 \end{bmatrix} \xrightarrow{\Sigma_j} \begin{bmatrix} X^\phi \\ 0 \end{bmatrix}$	$\begin{bmatrix} Z^\epsilon & F^\epsilon \\ V & 0 \end{bmatrix} \xrightarrow[\Sigma_i^\top]{\Sigma_j} \begin{bmatrix} X^\epsilon \\ V_{\text{tot}} \end{bmatrix}$
(II) input intensities	$\begin{bmatrix} Z^\phi \\ V \end{bmatrix} \widehat{X}^{\phi-1} \stackrel{\text{def}}{=} \begin{bmatrix} a^{\phi\phi} \\ v^{\phi\phi} \end{bmatrix}$	$\begin{bmatrix} Z^\epsilon \\ V \end{bmatrix} \widehat{X}^{\epsilon-1} \stackrel{\text{def}}{=} \begin{bmatrix} a^{\epsilon\epsilon} \\ v^{\epsilon\epsilon} \end{bmatrix}$
(OI) output intensities	$\widehat{X}^{\phi-1} \begin{bmatrix} Z^\phi & F^\phi \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} b^{\phi\phi} & f^{\phi\phi} \end{bmatrix}$	$\widehat{X}^{\epsilon-1} \begin{bmatrix} Z^\epsilon & F^\epsilon \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} b^{\epsilon\epsilon} & f^{\epsilon\epsilon} \end{bmatrix}$

Table 5: (A) **Physical and monetary** data used for EEIO ; and (II),(OI) associated input and output intensities - also called technical coefficients and allocation coefficients.

B Environmental accounting with Input-Output Analysis: Basic Formalism

B.1 Physical and monetary tables

This appendix section details classical notation for physical and monetary IOTs. Table 5 details notation for classical physical and monetary IOTs, and associated technical and allocation coefficients on which environmental footprinting is based.

B.2 Environmental responsibility tables

In EEIO, every transaction carries dual environmental attributions: a *product footprint* (the impact embedded in the physical flow) and a *monetary-flow footprint* (the impact associated with the corresponding revenue flow). Table 6 details notation for environmental IOTs, defining how these footprints propagate through economic networks. Product-footprint and monetary-footprint IOTs mirror the structure of physical and monetary IOTs, respectively — with the difference that direct impacts E are taken as an additional input or output, that is propagated down to F or up to V.

B.3 Environmental models

Out of the four colored intensity cells in Table 6, the four classical environmental footprinting models can be built, as detailed in Table 7. For each imputation direction, there are:

- a Leontief formulation (from input intensities a) and a Ghosh formulation (from output intensities b): the two rows
- an *intensity formulation* (lowercase x and e) and an *absolute-value formulation* (capital X and E)

The mapping between rows and formulations is direction-dependent: product footprints take intensity from row (II) and absolute-value from row (OI); monetary-flow footprints do the reverse. Both formulations within each column are equivalent, as detailed in C.3.

For maximum clarity, we now re-derive these four environmental models, with careful unit choice adapted to μ IOA (physical units for downstream imputation, monetary units for upstream). Note that they parallel the four basic input-output formulations identified in the literature [39].

Environmental units (ε)		
(p) Product embedded footprints	(m) Monetary flow embedded footprints	
(A) abs. values	$T^{\varepsilon P} = \begin{bmatrix} P & F^\varepsilon \\ 0 & 0 \\ E^\top & 0 \end{bmatrix} \xrightarrow{\sum_j} \begin{bmatrix} X^{\varepsilon P} \\ 0 \\ E_{\text{tot}} \end{bmatrix}$ $\downarrow \sum_i$ $\begin{bmatrix} X^{\varepsilon P \top} & E_{\text{tot}} \end{bmatrix}$	$T^{\varepsilon m} = \begin{bmatrix} M & 0 & E \\ V^\varepsilon & 0 & 0 \end{bmatrix} \xrightarrow{\sum_j} \begin{bmatrix} X^{\varepsilon m} \\ E_{\text{tot}} \end{bmatrix}$ $\downarrow \sum_i$ $\begin{bmatrix} X^{\varepsilon m \top} & 0 & E_{\text{tot}} \end{bmatrix}$
(II) input intensities	$\begin{bmatrix} P \\ 0 \\ E^\top \end{bmatrix} \widehat{X}^{\varepsilon P^{-1}} \stackrel{\text{def}}{=} \begin{bmatrix} p^{\varepsilon \phi} \\ 0 \\ e^{\varepsilon \phi \top} \end{bmatrix}$ $\downarrow \sum_i$ $x^{\varepsilon \phi P \top}$	$\begin{bmatrix} M \\ V^\varepsilon \end{bmatrix} \widehat{X}^{\varepsilon m^{-1}} \stackrel{\text{def}}{=} \begin{bmatrix} m^{\varepsilon \varepsilon} \\ v^{\varepsilon \varepsilon} \end{bmatrix}$
(OI) output intensities	$\widehat{X}^{\varepsilon P^{-1}} [P \quad F^\varepsilon] \stackrel{\text{def}}{=} [p^{\varepsilon \varepsilon} \quad f^{\varepsilon \varepsilon}]$	$\widehat{X}^{\varepsilon m^{-1}} [M \quad 0 \quad E] \stackrel{\text{def}}{=} [m^{\varepsilon \varepsilon} \quad 0 \quad e^{\varepsilon \varepsilon}] \xrightarrow{\sum_j} x^{\varepsilon \varepsilon m}$

Table 6: (A) Knowing physical and monetary data (Table 5) and direct footprints E, EEIO propagates E taken as an input (resp. output) through intermediate transactions with product footprints P (resp. monetary footprints M) down to final demand footprints F^ε (resp. up to value-added footprints F^ε). Zeros on the value-added row (resp. final demand column) reflect the downstream (resp. upstream) imputation objective; (II/OI) To do, based on input intensities (II) and output intensities (OI), Leontief and Ghosh models are applied, as detailed in Table 2.

	(p) Product embedded footprints	(m) Monetary flow embedded footprints
(II) Leontief: from input intensities a	$[\varepsilon \phi - p]$ $\text{Hyp: } p^{\varepsilon \phi} = \widehat{x}^{\varepsilon \phi P} a^{\phi \phi}$ $\Rightarrow x^{\varepsilon \phi P \top} (I - a^{\phi \phi}) = e^{\varepsilon \phi \top}$ $F^\varepsilon = x^{\varepsilon \phi P} \odot F^\phi$	$[\varepsilon - m]$ $\text{Hyp: } M = a^{\varepsilon \varepsilon} \widehat{X}^{\varepsilon m}$ $\Rightarrow X^{\varepsilon m} (I - a^{\varepsilon \varepsilon}) = E$ $V^\varepsilon = X^{\varepsilon m \top} \odot v^{\varepsilon \varepsilon}$
(OI) Ghosh: from output intensities b	$[\varepsilon - p]$ $\text{Hyp: } P = \widehat{X}^{\varepsilon P} b^{\phi \phi}$ $\Rightarrow X^{\varepsilon P \top} (I - b^{\phi \phi}) = E^\top$ $F^\varepsilon = X^{\varepsilon P} \odot f^{\phi \phi}$	$[\varepsilon \varepsilon - m]$ $\text{Hyp: } m^{\varepsilon \varepsilon} = b^{\varepsilon \varepsilon} \widehat{x}^{\varepsilon \varepsilon m}$ $\Rightarrow x^{\varepsilon \varepsilon m} (I - b^{\varepsilon \varepsilon}) = e^{\varepsilon \varepsilon}$ $V^\varepsilon = x^{\varepsilon \varepsilon m \top} \odot V$

Table 7: The four classic EEIO formulations, with units adapted to μ IOA. Each imputation direction (p vs m) has two *equivalent* formulations: one from input intensities (Leontief) and one from output intensities (Ghosh). Colors represent whether the main unknown to solve for is expressed in *intensity* (x) or *absolute-value* (X).

C Detailed formulations of the four environmental footprint calculation models

We saw in 3.2 and Table 2 that each type of environmental impact—(product footprints and monetary-flow footprints)— can be expressed as either *intensities* or *absolute values*. For brevity, the main text of this paper presents only intensity formulations, and does so in a minimal way, as these are well-known models. For

completeness, we now provide detailed derivations for all forms.

For reasons explained in 4.5, in μ IOA Leontief and Ghosh inverses $(I - a^{\phi\phi})^{-1}$ and $(I - b^{\phi\phi})^{-1}$ are usually undefined. In this appendix focused on the standard bidirectional EEIO formalism, we assume that they are anyway: this simplifies notation—allowing for example to denote the two steps of Table 2 left in one as in Equation 16— and does not hurt as all formulations remain applicable as long as the linear system is solvable.

C.1 Product footprints, intensity formulation $[\varepsilon\phi - p]$

By definition of product environmental intensities $x^{\varepsilon\phi P}$, we have the following balance vertically, as readable in cell (II)-(p) of Table 6:

$$1^\top p^{\varepsilon\phi} + e^{\varepsilon\phi\top} = x^{\varepsilon\phi P\top}$$

Environmental contents of inputs add-up ‘like energy content in a cooking recipe’: environmental content of products (per unit output) $p^{\varepsilon\phi}$ ¹¹ are calculated from environmental content of ingredients (per unit input) $x^{\varepsilon\phi P}$, weighted by required input quantities per output production $a^{\phi\phi}$:

$$p^{\varepsilon\phi} = \widehat{x^{\varepsilon\phi P} a^{\phi\phi}}$$

Replacing this in the balance equation and rearranging (assuming non-singular $I - a^{\phi\phi}$) yields product environmental intensities:

$$x^{\varepsilon\phi P\top} = e^{\varepsilon\phi\top} (I - a^{\phi\phi})^{-1} = e^{\varepsilon\phi\top} \ell^{\phi\phi} = 1^\top (e^{\varepsilon\phi} \odot \ell^{\phi\phi})$$

This is the classic Leontief intensity formulation for downstream imputation [24, 34]. It follows the form of the classical economic cost-push price dual [39] of the Leontief quantity model, substituting input environmental intensities for primary input prices.

Imputation to final demand We can use this intensity equation to calculate *embedded footprints* F^ε of final demand products (or equivalently, its transpose $F^{\varepsilon\top}$ ¹²) - here \odot denotes the Hadamard (element-wise) product:

$$F^{\varepsilon\top} = x^{\varepsilon\phi P\top} \odot F^{\phi\top} = x^{\varepsilon\phi P\top} \widehat{F^\phi}$$

$$F^{\varepsilon\top} = e^{\varepsilon\phi\top} \ell^{\phi\phi} \widehat{F^\phi} = F^{\phi\top} \widehat{e^{\varepsilon\phi\top} \ell^{\phi\phi}}$$

or, equivalently:

$$F^{\varepsilon\top} = 1^\top (e^{\varepsilon\phi} \odot \ell^{\phi\phi} \odot F^{\phi\top}) \tag{16}$$

This quantifies the environmental impact everywhere (summed on all producers) caused by every component of final demand. The calculation *imputes* the ‘responsibility’ of direct environmental impacts everywhere to the components of final demand, in the sense that the sum of final demand footprints equals the sum of direct impacts (E_{tot} in Table 6):

¹¹Note that product environmental intensity does not depend on buyer: all columns of $p^{\varepsilon\phi}$ are equal.

¹²maybe more intuitive than F^ε because it avoids transposing $L^{\phi\phi}$

$$\mathbf{F}^{\varepsilon\top} \mathbf{1} = \mathbf{1}^\top (\mathbf{e}^{\varepsilon\phi} \odot \ell^{\phi\phi} \odot \mathbf{F}^{\phi\top}) \mathbf{1} = \mathbf{1}^\top (\mathbf{E} \odot \mathbf{X}^{\phi(-1)} \odot \ell^{\phi\phi} \mathbf{F}^\phi) = \mathbf{1}^\top \mathbf{E} \quad (17)$$

Embedded product footprints of intermediate transactions follow in a similar way:

$$\mathbf{P}^\top = \mathbf{Z}^{\phi\top} \widehat{\mathbf{e}^{\varepsilon\phi\top} \ell^{\phi\phi}}$$

or written differently:

$$\mathbf{P} = \widehat{\mathbf{e}^{\varepsilon\phi\top} \ell^{\phi\phi}} \mathbf{Z}^\phi \quad (18)$$

Producer-disaggregated formulation The intensity equation 16 can also be turned into a producer interpretation, expressing the direct environmental impact *at each producer* required to meet the production (specified in terms of final demand or total).

$$\mathbf{E}^\top = \widehat{\mathbf{e}^{\varepsilon\phi\top} \mathbf{X}^\phi} = \widehat{\mathbf{e}^{\varepsilon\phi} \mathbf{X}^\phi}$$

$$\mathbf{E}^\top = \widehat{\mathbf{e}^{\varepsilon\phi}} \ell^{\phi\phi} \mathbf{F}^\phi = (\mathbf{e}^{\varepsilon\phi} \odot \ell^{\phi\phi} \odot \mathbf{F}^{\phi\top}) \mathbf{1} \quad (19)$$

Equations 16 and 19 have very similar forms but a crucial difference that results in them calculating very different quantities. By summing either on rows or columns of $(\mathbf{e}^{\varepsilon\phi} \odot \ell^{\phi\phi} \odot \mathbf{F}^{\phi\top})$, we focus the calculation either on the *origin* (producers) or on the *destination* (products) of the environmental impact flows, in the sense of table (A)-(T)-p.

We can also keep footprint calculations disaggregated *on both origin and destination*:

$$\mathbf{F}^{\varepsilon\top+} = \widehat{\mathbf{e}^{\varepsilon\phi}} \widehat{\ell^{\phi\phi} \mathbf{F}^\phi} = \widehat{\mathbf{e}^{\varepsilon\phi} \odot \ell^{\phi\phi} \odot \mathbf{F}^{\phi\top}} \quad (20)$$

each element at position i, j in this matrix represents the direct environmental impact at *producer* i imputed to final demand product j . Keeping a disaggregated account by origin can be useful when accounting for impacts that have a notion of *locality*—such as water use, or (partially-aggregated) for distinguishing data sources of various qualities.

Sum equality We have seen earlier in Equation 17 that the method imputes total direct impact to final demand components, verifying the sum equality $\mathbf{1}^\top \mathbf{F}^\varepsilon = \mathbf{E}^\top \mathbf{1}$.

Replacing $\mathbf{F}^\varepsilon = \widehat{\mathbf{x}^{\varepsilon\phi\mathbf{P}}} \mathbf{F}^\phi$ and $\mathbf{E}^\top = \widehat{\mathbf{e}^{\varepsilon\phi\top} \mathbf{X}^\phi}$ yields the classic sum equality:

$$\mathbf{x}^{\varepsilon\phi\mathbf{P}\top} \mathbf{F}^\phi = \mathbf{e}^{\varepsilon\phi\top} \mathbf{X}^\phi \quad (21)$$

C.2 Product footprints, **absolute-value formulation**

The vertical balance equation on $\mathbf{T}^{\varepsilon\mathbf{P}}$ (Table 6, upper left) is:

$$\mathbf{1}^\top \mathbf{P} + \mathbf{E}^\top = \mathbf{X}^{\varepsilon\mathbf{P}\top}$$

Here, the environmental content passed from a producer to its buyers is proportional to the physical quantity each buyer purchases, as given by the output coefficients:

$$P = \widehat{X^{\varepsilon P} b^{\phi\phi}}$$

Substituting into the balance equation (assuming $I - b^{\phi\phi}$ is non-singular) yields:

$$X^{\varepsilon P \top} = E^{\top} (I - b^{\phi\phi})^{-1} = E^{\top} g^{\phi\phi}$$

This is the Charpentier formulation [32] of product environmental contents—sometimes called ‘Impact Inheritance’—expressed in absolute environmental units. It follows the form of the classical Ghosh quantity model [39], substituting direct environmental impacts for primary input quantities.

Imputation interpretation We can use this equation to calculate *embedded footprints* F^{ε} of final demand products (or equivalently, its transpose $F^{\varepsilon \top}$), knowing $f^{\varepsilon\varepsilon} = f^{\phi\phi}$:

$$F^{\varepsilon \top} = (\widehat{X^{\varepsilon P} f^{\phi\phi}})^{\top} = (\widehat{f^{\phi\phi} X^{\varepsilon P}})^{\top} = X^{\varepsilon P \top} \widehat{f^{\phi\phi}}$$

$$F^{\varepsilon \top} = E^{\top} g^{\phi\phi} \widehat{f^{\phi\phi}} = f^{\phi\phi \top} \widehat{E^{\top} g^{\phi\phi}} \quad (22)$$

or, equivalently:

$$F^{\varepsilon \top} = 1^{\top} (E \odot g^{\phi\phi} \odot f^{\phi\phi \top}) \quad (23)$$

which is of course the same as Equation 16 - see C.3. Again, this formulation *imputes* the ‘responsibility’ of direct environmental impacts everywhere to final demand components.

Embedded product footprints of intermediate transactions are calculated in the same way, knowing $p^{\varepsilon\varepsilon} = b^{\phi\phi}$:

$$P^{\top} = b^{\phi\phi \top} \widehat{E^{\top} g^{\phi\phi}}$$

or written differently:

$$P = \widehat{E^{\top} g^{\phi\phi} b^{\phi\phi}}$$

Producer-disaggregated interpretation In a similar fashion as in C.1, in Equation 23 we can also avoid summing over origins E , in order to keep footprint calculation disaggregated on *both origin and destination*:

$$F^{\varepsilon \top +} = \widehat{E} g^{\phi\phi} \widehat{f^{\phi\phi}} = E \odot g^{\phi\phi} \odot f^{\phi\phi \top} \quad (24)$$

which are the same values as in Equation 20.

C.3 Product footprints: Equivalence between intensity and absolute value formulations

The equivalence between these two formulations of product environmental contents ($F^{\varepsilon\top} = e^{\varepsilon\phi\top} \ell^{\phi\phi} \widehat{F}^{\phi} = E^{\top} g^{\phi\phi} \widehat{f}^{\phi\phi}$) is shown using the relationship between $a^{\phi\phi} = Z^{\phi} \widehat{X}^{\phi-1}$ and $b^{\phi\phi} = \widehat{X}^{\phi-1} Z^{\phi}$:

$$a^{\phi\phi} \widehat{X}^{\phi} = \widehat{X}^{\phi} b^{\phi\phi}$$

We start from the Leontief intensity formulation and work up to the absolute value equivalence:

$$\begin{aligned} x^{\varepsilon\phi p\top} &= e^{\varepsilon\phi\top} (I - a^{\phi\phi})^{-1} \\ &= E^{\top} \widehat{X}^{\phi-1} (I - a^{\phi\phi})^{-1} \\ &= E^{\top} ((I - a^{\phi\phi}) \widehat{X}^{\phi})^{-1} \\ &= E^{\top} (\widehat{X}^{\phi} - Z^{\phi})^{-1} \end{aligned}$$

Right-multiply by $I = \widehat{X}^{\phi} \widehat{X}^{\phi-1}$ and reorganize:

$$\begin{aligned} x^{\varepsilon\phi p\top} &= E^{\top} (\widehat{X}^{\phi} - Z^{\phi})^{-1} \widehat{X}^{\phi} \widehat{X}^{\phi-1} \\ &= E^{\top} (\widehat{X}^{\phi-1} (\widehat{X}^{\phi} - Z^{\phi}))^{-1} \widehat{X}^{\phi-1} \\ &= E^{\top} (I - b^{\phi\phi})^{-1} \widehat{X}^{\phi-1} \end{aligned}$$

Right-multiply by \widehat{X}^{ϕ} and we find the Charpentier absolute value formulation:

$$X^{\varepsilon p\top} = x^{\varepsilon\phi p\top} \widehat{X}^{\phi} = E^{\top} (I - b^{\phi\phi})^{-1}$$

C.4 Monetary flow footprints, **intensity formulation** [$\varepsilon\text{€} - m$]

By definition of monetary flow environmental intensities $x^{\varepsilon\text{€}m}$, we have the following balance horizontally, as readable in cell (OI)-(m) of Table 6:

$$m^{\varepsilon\text{€}} 1 + e^{\varepsilon\text{€}} = x^{\varepsilon\text{€}m}$$

Again, environmental contents of incoming monetary flows add-up ‘like a cooking recipe’: environmental intensity of entities’ payments $r^{\varepsilon\text{€}}$ ¹³ are calculated from environmental intensity of their revenues $x^{\varepsilon\text{€}m}$, weighted by shares $b^{\varepsilon\text{€}}$ of each revenue origin in total revenue.

$$m^{\varepsilon\text{€}} = b^{\varepsilon\text{€}} \widehat{x^{\varepsilon\text{€}m}}$$

Replacing this in the balance equation and rearranging (assuming non-singular $I - b^{\varepsilon\text{€}}$) yields monetary flow environmental intensities:

¹³Note that monetary-flow environmental intensity from an entity does not depend on the recipient: all rows of $r^{\varepsilon\text{€}}$ are equal.

$$\mathbf{x}^{\varepsilon\in m} = (\mathbf{I} - \mathbf{b}^{\in\in})^{-1} \mathbf{e}^{\varepsilon\in} = \mathbf{g}^{\in\in} \mathbf{e}^{\varepsilon\in} = (\mathbf{g}^{\in\in} \odot \mathbf{e}^{\varepsilon\in\top}) \mathbf{1}$$

This is the Ghosh intensity formulation for upstream imputation. It follows the form of the classical economic revenue-pull price dual [39] of the Ghosh quantity model, substituting output environmental intensities for final demand product prices. While the original supply-driven model dates to Ghosh [2], an early explicit formulation of its application to environmental responsibility is given by [11].

Imputation to value added We can use this intensity equation to calculate *embedded footprints* V^ε of value-added revenue (or equivalently, its transpose $V^{\varepsilon\top}$ ¹⁴)

$$V^{\varepsilon\top} = \mathbf{x}^{\varepsilon\in m} \odot V^\top = \widehat{V} \mathbf{x}^{\varepsilon\in m}$$

$$V^{\varepsilon\top} = \widehat{V} \mathbf{g}^{\in\in} \mathbf{e}^{\varepsilon\in} = \widehat{\mathbf{g}^{\in\in} \mathbf{e}^{\varepsilon\in}} V^\top$$

or equivalently,

$$V^{\varepsilon\top} = (V^\top \odot \mathbf{g}^{\in\in} \odot \mathbf{e}^{\varepsilon\in\top}) \mathbf{1} \quad (25)$$

This quantifies the environmental impact everywhere (summed on all producers) caused by every component of value added. The calculation *imputes* the ‘responsibility’ of direct environmental impacts everywhere to the components of value added, in the sense that the sum of value added footprints equals the sum of direct impacts (E_{tot} in Table 6):

$$\mathbf{1}^\top V^{\varepsilon\top} = \mathbf{1}^\top (V^\top \odot \mathbf{g}^{\in\in} \odot \mathbf{e}^{\varepsilon\in\top}) \mathbf{1} = (V \mathbf{g}^{\in\in} \odot X^{\in(-1)} \odot E^\top) \mathbf{1} = E^\top \mathbf{1} \quad (26)$$

Embedded monetary flow footprints of intermediate transactions follow in a similar way:

$$M^\top = \widehat{\mathbf{g}^{\in\in} \mathbf{e}^{\varepsilon\in}} Z^{\varepsilon\top}$$

or written differently:

$$M = Z^{\varepsilon\in} \widehat{\mathbf{g}^{\in\in} \mathbf{e}^{\varepsilon\in}} \quad (27)$$

Producer-disaggregated formulation Again, in Equation 25 we can also avoid summing over origins $\mathbf{e}^{\varepsilon\in}$, in order to keep footprint calculation disaggregated on *both origin and destination*:

$$V^{\varepsilon\top+} = \widehat{V} \widehat{\mathbf{g}^{\in\in} \mathbf{e}^{\varepsilon\in}} = V^\top \odot \mathbf{g}^{\in\in} \odot \mathbf{e}^{\varepsilon\in\top} \quad (28)$$

Sum equality We have seen earlier in Equation 26 that the method imputes total direct impact to value-added components, verifying the sum equality $V^{\varepsilon\top} \mathbf{1} = \mathbf{1}^\top E$.

Again, replacing $V^\varepsilon = V \mathbf{x}^{\varepsilon\in r}$ and $E = \mathbf{e}^{\varepsilon\in} X^{\varepsilon\in}$ yields the classic sum equality:

$$V \mathbf{x}^{\varepsilon\in r} = \mathbf{e}^{\varepsilon\in\top} X^{\varepsilon\in} \quad (29)$$

¹⁴maybe more intuitive than V^ε because it avoids transposing $G^{\in\in}$

C.5 Monetary flow footprints, absolute-value formulation

The horizontal balance equation on $T^{\varepsilon m}$ (Table 6, upper right) is:

$$M1 + E = X^{\varepsilon m}$$

Here, the environmental content passed from producers to their suppliers is proportional to the monetary quantity paid suppliers, as given by the monetary-unit input coefficients:

$$M = a^{\varepsilon\varepsilon} \widehat{X^{\varepsilon m}}$$

Substituting into the balance equation (assuming $I - a^{\varepsilon\varepsilon}$ is non-singular) yields:

$$X^{\varepsilon m} = (I - a^{\varepsilon\varepsilon})^{-1} E = \ell^{\varepsilon\varepsilon} E$$

This is the upstream counterpart of the Charpentier absolute-value formulation, focused on monetary flow environmental contents instead of product environmental contents. It follows the form of the classical Leontief quantity model [39], substituting direct environmental impacts for final demand quantities.

Imputation interpretation We can use this equation to calculate *embedded footprints* V^{ε} of value-added revenue (or equivalently, its transpose $V^{\varepsilon T}$), knowing $v^{\varepsilon\varepsilon} = v^{\varepsilon\varepsilon\varepsilon}$:

$$V^{\varepsilon T} = (v^{\varepsilon\varepsilon} \widehat{X^{\varepsilon m}})^T = (X^{\varepsilon m T} \widehat{v^{\varepsilon\varepsilon}})^T = \widehat{v^{\varepsilon\varepsilon}} X^{\varepsilon m}$$

$$V^{\varepsilon T} = \widehat{v^{\varepsilon\varepsilon}} \ell^{\varepsilon\varepsilon} E = \widehat{\ell^{\varepsilon\varepsilon}} E v^{\varepsilon\varepsilon T} \quad (30)$$

or, equivalently:

$$V^{\varepsilon T} = (v^{\varepsilon\varepsilon T} \odot \ell^{\varepsilon\varepsilon} \odot E^T) 1 \quad (31)$$

which is of course the same as Equation 25. Again, this formulation to *imputes* the ‘responsibility’ of direct environmental impacts everywhere to value-added components.

Embedded monetary flow footprints of intermediate transactions are calculated in the same way, knowing $m^{\varepsilon\varepsilon} = a^{\varepsilon\varepsilon\varepsilon}$:

$$M^{\varepsilon T} = \widehat{\ell^{\varepsilon\varepsilon}} E a^{\varepsilon\varepsilon T}$$

or written differently:

$$M^{\varepsilon} = a^{\varepsilon\varepsilon} \widehat{\ell^{\varepsilon\varepsilon}} E$$

Producer-disaggregated interpretation Again, in Equation 31 we can also avoid summing on origins E , in order to keep footprint calculation disaggregated on *both origin and destination*:

$$V^{\varepsilon T+} = \widehat{v^{\varepsilon\varepsilon}} \ell^{\varepsilon\varepsilon} \widehat{E} = v^{\varepsilon\varepsilon T} \odot \ell^{\varepsilon\varepsilon} \odot E^T \quad (32)$$

which are the same values as in Equation 28.

C.6 Monetary flow footprints: Equivalence between intensity and absolute value formulations

Again, the intensity and absolute-value formulations of monetary flow environmental contents are equivalent.

The equivalence between them ($V^{\varepsilon\top} = \widehat{V} \quad g^{\varepsilon\varepsilon} e^{\varepsilon\varepsilon} = \widehat{v}^{\varepsilon\varepsilon} \ell^{\varepsilon\varepsilon} E$) is demonstrated in the same way as for product footprints: see C.3 - we do not repeat the demonstration.

D Product Classification and Granularity in μ IOA

Defining the ‘Product’ in a Micro-scale Context In the μ IOA formalism, a ‘product’ is defined as any distinct material or service flow. This includes not only market-traded commodities but also internal intermediates that never leave the entity’s boundary. Unlike traditional EEIO, which is bound by rigid national classification codes (e.g., NACE), the μ IOA architecture allows the product classification to be determined dynamically based on available data.

Technological Substitutability: The Granularity Threshold The highest precision in μ IOA footprints is achieved when the product classification reflects the *technological substitutability* of inputs within production processes. For instance, if two batches of *304 stainless steel sheet* from different suppliers are used interchangeably in all downstream processes, they can be treated as a single product. Conversely, if flour grade A is used exclusively for bread and grade B for pastries, they are *technologically distinct*. Maximally granular footprints are obtained when these streams are classified separately.

Graceful Degradation and Inherent Robustness A core strength of the μ IOA methodology is its “graceful degradation” under coarse or noisy data (e.g., inconsistent e-invoicing labels):

- *Operational Continuity*: If distinct grades are merged into a single category (e.g., aggregating grades A and B into “Flour”), the framework remains fully operational. Downstream products simply inherit the environmental intensity of the resulting *composite input*—representing the finest granularity permitted by the available data.
- *Entity-Product Separability*: A coarse classification (e.g., merging ‘Iron Ore’ and ‘Scrap’ into ‘Iron Materials’) only impacts the final footprints if a single producer *sells both items* under that merged label. Because μ IOA treats every {entity, product} pair as a unique node, as long as these two products are produced by distinct entities, the system naturally maintains their unique environmental profiles regardless of the shared label.

Implementation and Standardization Pathways The product classification need not be predefined; it emerges ad hoc from the internal process declarations and e-invoice labels managed by firm accountants. This flexibility allows for an iterative refinement of data resolution without imposing a universal taxonomy. By focusing standardisation efforts on high-impact categories—such as high-emission commodities or materials under trade scrutiny (e.g., CBAM)—this targeted and progressive approach ensures maximal footprinting precision with minimal product labeling effort.

E A Phased, Value-Driven Integration Outlook

A fully realized μ IOA system—with firms reporting unit-process inventories (S^k) and centralized allocation rules (Θ^p)—need not be built in a single step. This appendix is a slightly longer description of Section 7.1.2 : how we envision μ IOA could be phased-in in practice. It is intended as a pragmatic example—certainly not a definitive plan—for practical implementation, growing in accuracy and delivering value at each step without requiring a monolithic upfront investment.

Phase 1: Prototype with approximated Π . Launch by pairing observed transaction data Ω (from e-invoicing) with an *approximated* Π without declaration of the S^k , as described in 7.1.2. Multiple approximation methods are possible; for instance, Π can be initialized from a sector-average process template (e.g., derived from LCA databases [22]) and then scaled or adjusted to align with the entity’s observed total inputs and outputs from Ω . This automated prototype already delivers micro-scale footprints with far greater completeness and firm-level specificity than current aggregated or truncated methods, demonstrating feasibility and building stakeholder engagement.

Phase 2.1: Expand with benchmark process data. Systematically improve Π by collecting *actual unit process inventories* (S^k) from a carefully sampled subset of entities (e.g., industry leaders, large emitters). Use these high-quality benchmarks to derive and validate *centralized allocation rules* (Θ^P). These benchmarks can also serve as training data for statistical or machine-learning models that predict plausible process structures for other entities based on observable attributes such as sector, size, and input or output portfolios.

Phase 2.2: Refine through targeted data collection. Use the improved μ IOA model to establish a *feedback loop for targeted refinement*. The system itself identifies *high-leverage hotspots*—such as the largest flows of carbon and materials, the most uncertain processes, or areas under acute regulatory scrutiny (e.g., CBAM). Resources for ground-truth data collection and auditing are then focused on these priority areas, which in turn further refines the model’s accuracy and reliability where it matters most.

Phase 3: Institutionalize with standardized reporting. Align μ IOA data requirements with evolving regulatory frameworks for supply-chain transparency (e.g., CSDDD [38]), digital product passports [35], and corporate sustainability reporting (CSRD [41]). This integration creates a *virtuous cycle*: regulation mandates standardized reporting of transactions (Ω) and unit-process data (S^k); the resulting micro-scale data feeds more accurate, actionable footprints; and the improved footprints, in turn, support more effective and equitable policy design and enforcement.

F Toy Micro-Economy

F.1 Toy Micro-Economy: Graph Definition

This appendix provides the complete specification of the toy economy used in Section 5: unit process recipes, sourcing preferences, constraint parameters, initialization values, target ranges (for sourcings, markups, final demand, prices, and premium ratios), and the optimization weights assigned to each sub-loss.

```
1 #####
2 # COMPONENTS and CONNECTIONS
3 #####
4
5 process_recipes = {
6 # --- Power Generation ---
7 'PowerGen_Green': {
8   'outputs': {'E': 1.0},
9   'output_type': {'E': 'external'},
10  'environment': {'CO2': 50}, # Low-carbon: 50 kg CO2 per MWh
11  # for this simple example we monitor only 1 env impact, but could easily extend
12  'init_operating_costs': 80 # compensation of employees (wages, benefits)
13  # for this simple example we keep labor vs capital together, but could easily separate them in a
14  # real application.
15 },
16 'PowerGen_Brown': {
17   'outputs': {'E': 1.0},
18   'output_type': {'E': 'external'},
19   'environment': {'CO2': 800}, # High-carbon: 800 kg CO2 per MWh
20   'init_operating_costs': 40
21 },
22 # --- Mining ---
23 'IronOreExtraction': {
24   'outputs': {'IronOre': 1.0},
25   'output_type': {'IronOre': 'external'},
26   'inputs': {'E': 0.1}, # 0.1 MWh per ton ore
27   'input_type': {'E': 'external'},
28   'environment': {'CO2': 100}, # 100 kg CO2 per ton ore
29   'init_operating_costs': 30
30 },
31 # --- Steel Production ---
32 'SteelMake_BF': { # Blast furnace (ore-based)
33   'outputs': {'S': 1.0},
34   'output_type': {'S': 'external'},
35   'inputs': {'E': 0.5, 'IronOre': 1.5},
36   'input_type': {'E': 'external', 'IronOre': 'external'},
37   'environment': {'CO2': 1950}, # High direct environment: 1950 kg CO2 per ton
38   'init_operating_costs': 150
39 },
40 'SteelMake_EF': { # Electric arc furnace (scrap-based)
41   'outputs': {'S': 1.0},
42   'output_type': {'S': 'external'},
43   'inputs': {'E': 4.0, 'Scrap': 1.05}, # More electricity, less environment
44   'input_type': {'E': 'external', 'Scrap': 'external'},
45   'environment': {'CO2': 50}, # Low direct environment: 50 kg CO2 per ton
46   'init_operating_costs': 200
47 },
48 # --- Manufacturing (Advanced) ---
49 'Stamping_Advanced': {
50   'outputs': {'C_Unfinished': 1.0, 'Scrap': 0.7},
51   'output_type': {'C_Unfinished': 'internal', 'Scrap': 'both'}, # C_Unfinished is internal
52   'theta': {'C_Unfinished': 0.99, 'Scrap': 0.01},
53   # Note: allocation ratios are the same for all inputs.
54   # If the maker entity has technological knowledge that allows differentiating them per input,
55   # then the process is not elementary => split it according to technology
56   'inputs': {'E': 10.0, 'S': 8 / 6},
57   'input_type': {'E': 'external', 'S': 'external'},
58   'environment': {'CO2': 100 / 6}, # ~16.7 kg CO2 per C_Unfinished
59   'init_operating_costs': 2000,
60 },
61 'Finishing_Advanced': {
62   'parent': 'Stamping_Advanced',
63   'parent_coupling': 1.0, # 1 Finishing per 1 Stamping
64   'outputs': {'C': 0.833}, # Yield: 5 cars from 6 C_Unfinished
65   'output_type': {'C': 'external'},
66   'inputs': {'C_Unfinished': 1.0, 'Scrap': 0.5 / 6, 'T': 40 / 6},
67   'input_type': {'C_Unfinished': 'internal', 'Scrap': 'internal', 'T': 'external'},
68 }
69 }
```

```

70     'environment': {'CO2': 50 / 6}, # ~8.3 kg CO2 per C_Unfinished processed
71     'init_operating_costs': 4000,
72 },
73
74 # --- Manufacturing (Basic) ---
75 'Stamping_Basic': {
76     'outputs': {'C_Unfinished': 1.0, 'Scrap': 1.0},
77     'output_type': {'C_Unfinished': 'internal', 'Scrap': 'both'}, # C_Unfinished is internal
78     'theta': {'C_Unfinished': 0.96, 'Scrap': 0.04},
79     # vs advanced: we use a different allocation rule here to simulate a different allocation rule
80     # (mixed of physical, monetary, and other...)
81     'inputs': {'E': 16.0, 'S': 2.0, 'C': 0.05},
82     # REM: Stamping_Basic uses a bit of 'basic cars C' from 'Finishing_Basic' to transport parts...
83     # This is useful to show an internal loop ((I-a)^{-1} computation with non-zero a in internal
84     #       alloc)
85     'input_type': {'E': 'external', 'S': 'external', 'C': 'internal'},
86     'environment': {'CO2': 100},
87     'init_operating_costs': 1000
88 },
89 'Finishing_Basic': {
90     'parent': 'Stamping_Basic',
91     'parent_coupling': 1.0, # 1 Finishing per 1 Stamping
92     'outputs': {'C': 0.8}, # Yield: 4 cars from 5 C_Unfinished
93     'output_type': {'C': 'both'},
94     'inputs': {'C_Unfinished': 1.0, 'Scrap': 0.8 / 5, 'T': 30 / 5},
95     'input_type': {'C_Unfinished': 'internal', 'Scrap': 'internal', 'T': 'external'},
96     'environment': {'CO2': 80 / 5},
97     'init_operating_costs': 2000,
98 },
99
100 # --- Services ---
101 'ServiceProvision': {
102     'outputs': {'T': 1.0},
103     'output_type': {'T': 'external'},
104     'inputs': {'E': 0.27}, # 0.27 MWh per service-hour
105     'input_type': {'E': 'external'},
106     'environment': {'CO2': 200 / 170}, # ~1.18 kg CO2 per service-hour
107     'init_operating_costs': 120
108 }
109
110 # ===== ENTITY-PROCESS MAPPING =====
111 # List of (entity, process) pairs - defines both entities and their processes
112 # In this simple example each process is operated by max 1 entity ;
113 # but it would be no problem to relax
114 ENTITY_PROCESS_PAIRS = [
115     ('PowerCo_Green', 'PowerGen_Green'),
116     ('PowerCo_Brown', 'PowerGen_Brown'),
117     ('MiningCo', 'IronOreExtraction'),
118     ('SteelCo_BF', 'SteelMake_BF'),
119     ('SteelCo_EF', 'SteelMake_EF'),
120     ('ManuCo_Advanced', 'Stamping_Advanced'),
121     ('ManuCo_Advanced', 'Finishing_Advanced'),
122     ('ManuCo_Basic', 'Stamping_Basic'),
123     ('ManuCo_Basic', 'Finishing_Basic'),
124     ('ServiceCo', 'ServiceProvision'),
125 ]
126
127 ENTITIES = sorted({entity for entity, _ in ENTITY_PROCESS_PAIRS})
128 ALL_ENTITIES_WITH_FINAL = ENTITIES + ['FINAL']
129
130 EEIO_SECTORS = {
131     'Energy': ['PowerCo_Green', 'PowerCo_Brown'],
132     'Materials': ['MiningCo', 'SteelCo_BF', 'SteelCo_EF'],
133     'Manufacturing': ['ManuCo_Advanced', 'ManuCo_Basic'],
134     'Services': ['ServiceCo']
135 }
136 EEIO_COMPANY_TO_SECTOR_MAPPING = {company: sector for sector, companies in EEIO_SECTORS.items() for
137     company in companies}
138
139 # ===== PRODUCT DEFINITIONS =====
140 ALL_PRODUCTS = {
141     'E': 'Electricity', # (produced by Green AND Brown)
142     'S': 'Steel', # (produced by BF AND EF)
143     'IronOre': 'IronOre',
144     'C': 'Finished_Car_Body',
145     'T': 'Transport_Design_Service',
146     'Scrap': 'Scrap_Metal', # (traded + internal)
147     'C_Unfinished': 'Unfinished_Car_Body', # Purely internal intermediate
148 }

```

```

149 ENVIRONMENT = {
150   'CO2': 'Carbon_Dioxide'
151 }
152
153
154 ALL_UNITS = {
155   'E': 'MWh',
156   'S': 'ton',
157   'IronOre': 'ton',
158   'Scrap': 'ton',
159   'C': 'unit',
160   'C_Unfinished': 'unit',
161   'T': 'serv-h',
162   'CO2': 'kg_CO2-eq',
163   'labor_capital': 'euro'
164 }
165
166 #####
167 # INITIAL VALUES
168 #####
169
170 # ===== INITIAL PROCESS SCALES =====
171 # Reasonable starting guesses for optimization
172 INITIAL_SCALES = {
173   ('PowerCo_Green', 'PowerGen_Green'): 60.0,
174   ('PowerCo_Brown', 'PowerGen_Brown'): 40.0,
175   ('MiningCo', 'IronOreExtraction'): 15.0,
176   ('SteelCo_BF', 'SteelMake_BF'): 8.0,
177   ('SteelCo_EF', 'SteelMake_EF'): 8.0,
178   ('ManuCo_Advanced', 'Stamping_Advanced'): 6.0,
179   ('ManuCo_Basic', 'Stamping_Basic'): 5.0,
180   ('ServiceCo', 'ServiceProvision'): 100.0,
181 }
182
183 # ===== INITIAL OPERATING COSTS =====
184 # defined directly in process_recipes
185
186 # ===== INITIAL PRICES =====
187 # determined dynamically: middle of price target ranges
188
189
190 #####
191 # CONSTRAINTS
192 #####
193
194 # ===== SOURCING PREFERENCES =====
195 # TARGET fractions at ENTITY LEVEL
196 # format: {product: {buyer_entity: {seller_entity: target_fraction  $\wedge$  pref[p][b][s]}}
197
198 # Underlying hypothesis so far: each entity has max 1 process that makes any output (never 2)
199 # => origin mapping can be per entity, and we know process origin
200
201 # REMARK: ENTITY_SOURCING_PREFERENCES also defines allowed transactions:
202 # e.g. if we set {'E': 'MiningCo': {'PowerCo_Brown': 1.0}}, there will be
203 # no PowerCo_Green to MiningCo connection in the graph.
204
205 ENTITY_SOURCING_PREFERENCES = {
206   'E': { # Electricity preferences
207     'SteelCo_BF': {'PowerCo_Brown': 0.7, 'PowerCo_Green': 0.3},
208     'SteelCo_EF': {'PowerCo_Green': 0.6, 'PowerCo_Brown': 0.4},
209     'ManuCo_Advanced': {'PowerCo_Green': 0.9, 'PowerCo_Brown': 0.1},
210     'ManuCo_Basic': {'PowerCo_Brown': 0.9, 'PowerCo_Green': 0.1},
211     'ServiceCo': {'PowerCo_Green': 0.6, 'PowerCo_Brown': 0.4},
212     'MiningCo': {'PowerCo_Brown': 1.0, 'PowerCo_Green': 1e-6},
213   },
214   'S': { # Steel preferences (KEY technological matching)
215     'ManuCo_Advanced': {'SteelCo_EF': 1.0, 'SteelCo_BF': 1e-6}, # Advanced wants EF steel
216     'ManuCo_Basic': {'SteelCo_BF': 1.0, 'SteelCo_EF': 1e-6}, # Basic wants BF steel
217   },
218   'Scrap': { # Scrap recycling preferences
219     'SteelCo_EF': {'ManuCo_Advanced': 0.6, 'ManuCo_Basic': 0.4}
220   },
221   'T': { # Services
222     'ManuCo_Advanced': {'ServiceCo': 1.0},
223     'ManuCo_Basic': {'ServiceCo': 1.0},
224   },
225   'IronOre': { # Iron ore
226     'SteelCo_BF': {'MiningCo': 1.0}
227   },
228 }
229

```

```

230 FINAL_SOURCING_PREFERENCES = {
231   'E': {'PowerCo_Green': 0.6, 'PowerCo_Brown': 0.4},
232   'S': {'SteelCo_BF': 0.5, 'SteelCo_EF': 0.5},
233   'T': {'ServiceCo': 1.0},
234   'C': {'ManuCo_Advanced': 0.6, 'ManuCo_Basic': 0.4}
235 }
236 # (also serves for defining which sourcings are allowed in FD)
237
238 # ===== MARKUP RANGES BY ENTITY TYPE =====
239
240 # TARGET_MARKUPS on TOTAL COSTS (Purchases + Operating_Costs)
241 # Format: {entity_pattern: (min_markup, max_markup)}
242 TARGET_MARKUPS = {
243   'PowerCo': (0.05, 0.08),      # Utilities: narrow range, regulated
244   'MiningCo': (0.12, 0.18),    # Commodity: moderate range
245   'SteelCo': (0.08, 0.15),     # Cyclical: medium range
246   'ManuCo_Advanced': (0.10, 0.18), # Specialized: moderate range
247   'ManuCo_Basic': (0.06, 0.12),  # Basic: narrow range
248   'ServiceCo': (0.12, 0.20),    # Services: moderate range
249 }
250
251 # ===== FINAL DEMAND CONSTRAINTS =====
252
253 TARGET_FINAL_DEMAND_RANGES = {
254   'E': (50, 80),      # 50-80 MWh electricity
255   'S': (2, 5),       # 2-5 tons steel
256   'C': (8, 10),     # 8-10 cars
257   'T': (80, 120),   # 80-120 service-hours
258 }
259 # This also defines which products are 'allowed' to have final demand
260
261 # ===== PRICE CONSTRAINTS =====
262
263 # Reasonable absolute price bounds (per product)
264 TARGET_PRICE_RANGES = {
265   'E': (50, 150),   # Electricity: 50-150 EUR/MWh
266   'S': (500, 1500), # Steel: 500-1500 EUR/ton
267   'IronOre': (80, 200), # Iron ore: 80-200 EUR/ton
268   'C': (15000, 35000), # Cars: 15k-35k EUR/unit
269   'T': (100, 250),  # Services: 100-250 EUR/hour
270   'Scrap': (100, 400), # Scrap: 200-400 EUR/ton
271 }
272 # This also defines price init values (init to middle-of-range values)
273
274 # --- SYSTEMATIC/EXPECTED DIFFERENCES: PREMIUM ratios ---
275 # Some producers are expected to have a premium on the price of (one of) their product
276 TARGET_PREMIUM_RATIOS = {
277   # Green electricity expected to be 15% more expensive than Brown
278   'E': {'PowerCo_Green': 1.15},
279   # EF steel expected to be 10% more expensive than BF
280   'S': {'SteelCo_EF': 1.10},
281   # Advanced cars expected to be 50% more expensive than Basic
282   'C': {'ManuCo_Advanced': 1.50}
283 }
284
285 # --- RANDOM/UNEXPLAINED VARIATION (market imperfections) ---
286 # After adjusting prices for expected differences (PRODUCER_COMPARISON_RATIOS),
287 # remaining price differences should be ≤ this tolerance
288 CROSS_PRODUCER_TOLERANCE = 0.03
289
290 # --- MIN OPERATING COSTS -----
291 # Factor by which we allow operating costs to be below their init value
292 # (defined in recipes) before penalizing
293 MIN_OP_COST_FACTOR = 3.
294
295 #####
296 # WEIGHTS USED FOR OPTIM #
297 #####
298
299 # ===== PREFERENCE WEIGHTS =====
300 # How strongly to enforce sourcing preferences for each (buyer_entity, product) pair
301 # Format: (product, buyer_entity): weight
302 PREFERENCE_WEIGHTS = {
303   # CRITICAL: Steel technological preferences
304   ('S', 'ManuCo_Advanced'): 1.0,
305   ('S', 'ManuCo_Basic'): 1.0,
306
307   # IMPORTANT: Manufacturers' electricity preferences
308   ('E', 'ManuCo_Advanced'): 0.5,
309   ('E', 'ManuCo_Basic'): 0.5,
310 }

```

```

311 # IMPORTANT: Scrap recycling
312 ('Scrap', 'SteelCo_EF'): 0.3,
313
314 # Final demand sourcing
315 ('E', 'FINAL'): 0.1,
316 ('S', 'FINAL'): 0.1,
317 ('C', 'FINAL'): 0.5,
318 ('T', 'FINAL'): 0.1,
319
320 # MEDIUM: Other electricity preferences
321 ('E', 'SteelCo_BF'): 0.1,
322 ('E', 'SteelCo_EF'): 0.1,
323 ('E', 'ServiceCo'): 0.1,
324 ('E', 'MiningCo'): 0.1,
325 }
326
327 DEFAULT_WEIGHT = 0.1 # For unspecified (buyer_entity, product) pairs
328
329 # ===== LOSS WEIGHTS (optimization priorities) =====
330 LOSS_WEIGHTS = {
331 # physical
332 'mat_balance_fd': 100000., # both for FD (<0 penalty) and non-FD (!=0 penalty) products
333 'mat_balance_nfd': 100000.,
334 'origin_pref_loss': 100.,
335 'in_range_fd_loss': 1000.,
336 # monetary
337 'pos_markup': 1000.,
338 'in_range_markup': 100.,
339 'price_consistency': 1000.,
340 'in_range_price': 100.,
341 'min_op_cost': 100.
342 }
343
344 # ===== REMARKS ABOUT PROCESS RECIPES =====
345 # Simple version:
346 # - processes get their inputs either totally from inside, or totally from outside
347 # - but process outputs can be simultaneously internal and external
348 #   (=partially, but not totally used by another internal process)
349
350 # In a more advanced version, we could:
351 # * have more advanced child/parent relationships (with soft couplings)
352 # * compute dynamically which outputs are tradable (not fully used by children) or not
353 # * compute dynamically which inputs are internal vs external
354 # But for this simple example, let's just hard-code these.
355
356 # Note: in a real-world system, recipes could be static (fixed physical process)
357 # or dynamic (steel can use any proportion of new vs recycled)

```

Listing 1: Micro-economy definition: data/core_data.py

F.2 Toy Micro-Economy: Balanced State

This appendix reports the final optimized state of the toy economy after the balancing procedure described in 6.2. It includes all process scales, physical transaction quantities, entity-level markups, and product prices, as well as the resulting sourcing fractions and final demand totals.

```

1 =====
2 FINAL OPTIMIZED STATE - pIOA Toy Economy
3 =====
4
5 =====
6 LOSSES
7 =====
8
9 PHYSICAL LOSSES:
10 Loss Type                               Unweighted   Weighted
11 -----
12 Material balance (non-FD)                0.000000     0.002083
13 Material balance (FD)                    0.000000     0.000000
14 Preference loss                           0.004384     0.438425
15 FD range loss                             0.000000     0.000000
16
17 MONETARY LOSSES:
18 Loss Type                               Unweighted   Weighted

```

```

19 -----
20 Positive markup loss          0.000000    0.000000
21 Markup range loss            0.000000    0.000000
22 Price Consistency loss       0.000000    0.000017
23 Price range loss             0.000000    0.000000
24
25 =====
26 OPERATING COSTS PER PROCESS (EUR per unit scale)
27 =====
28
29 ManuCo_Advanced:
30   Finishing_Advanced         : 9614.32 EUR/scale (min: 1333.33) × scale 6.540 = 62879.4 EUR total ✓
31   Stamping_Advanced         : 4807.17 EUR/scale (min: 666.67) × scale 6.540 = 31439.7 EUR total ✓
32
33 ManuCo_Basic:
34   Finishing_Basic           : 5122.10 EUR/scale (min: 666.67) × scale 3.762 = 19271.3 EUR total ✓
35   Stamping_Basic            : 2561.05 EUR/scale (min: 333.33) × scale 3.762 = 9635.6 EUR total ✓
36
37 MiningCo:
38   IronOreExtraction         : 107.09 EUR/scale (min: 10.00) × scale 17.647 = 1889.8 EUR total ✓
39
40 PowerCo_Brown:
41   PowerGen_Brown            : 69.27 EUR/scale (min: 13.33) × scale 103.617 = 7177.5 EUR total ✓
42
43 PowerCo_Green:
44   PowerGen_Green            : 78.05 EUR/scale (min: 26.67) × scale 165.320 = 12903.3 EUR total ✓
45
46 ServiceCo:
47   ServiceProvision          : 104.44 EUR/scale (min: 40.00) × scale 152.549 = 15932.9 EUR total ✓
48
49 SteelCo_BF:
50   SteelMake_BF              : 496.36 EUR/scale (min: 50.00) × scale 11.765 = 5839.5 EUR total ✓
51
52 SteelCo_EF:
53   SteelMake_EF              : 200.39 EUR/scale (min: 66.67) × scale 6.851 = 1372.9 EUR total ✓
54
55 =====
56 ENTITY MARKUPS
57 =====
58 ManuCo_Advanced      : 12.35% (target: 10.0-18.0%) ✓ pos:✓
59 ManuCo_Basic         : 6.84% (target: 6.0-12.0%) ✓ pos:✓
60 MiningCo             : 14.94% (target: 12.0-18.0%) ✓ pos:✓
61 PowerCo_Brown        : 5.83% (target: 5.0-8.0%) ✓ pos:✓
62 PowerCo_Green        : 6.95% (target: 5.0-8.0%) ✓ pos:✓
63 ServiceCo            : 16.38% (target: 12.0-20.0%) ✓ pos:✓
64 SteelCo_BF           : 12.58% (target: 8.0-15.0%) ✓ pos:✓
65 SteelCo_EF           : 12.89% (target: 8.0-15.0%) ✓ pos:✓
66
67 =====
68 PRICE ANALYSIS (EUR/unit)
69 =====
70
71 C (unit) [Target 15000-35000 EUR/unit]:
72   ManuCo_Advanced      : 23249.7 EUR/unit (1 buyer) ✓ [Premium: 1.45x vs 1.50x, diff: -3.0% ✓]
73   ManuCo_Basic         : 15981.3 EUR/unit (1 buyer) ✓
74
75 E (MWh) [Target 50-150 EUR/MWh]:
76   PowerCo_Brown        : 73.4 EUR/MWh (71.2-75.3) ✓
77   PowerCo_Green        : 84.6 EUR/MWh (82.8-86.7) ✓ [Premium: 1.15x vs 1.15x, diff: +0.3% ✓]
78
79 IronOre (ton) [Target 80-200 EUR/ton]:
80   MiningCo             : 131.7 EUR/ton (1 buyer) ✓
81
82 S (ton) [Target 500-1500 EUR/ton]:
83   SteelCo_BF           : 823.3 EUR/ton (812.5-831.3) ✓
84   SteelCo_EF           : 916.7 EUR/ton (912.5-922.3) ✓ [Premium: 1.11x vs 1.10x, diff: +1.2% ✓]
85
86 Scrap (ton) [Target 100-400 EUR/ton]:
87   ManuCo_Advanced      : 269.1 EUR/ton (1 buyer) ✓
88   ManuCo_Basic         : 274.8 EUR/ton (1 buyer) ✓
89
90 T (serv-h) [Target 100-250 EUR/serv-h]:
91   ServiceCo            : 146.1 EUR/serv-h (144.1-148.4) ✓
92
93 =====
94 FINAL DEMAND ANALYSIS
95 =====
96
97 FINAL DEMAND VALUES (positive = supply to FD customers, negative = forbidden):
98   PowerCo_Green → E      : 40.35849 MWh (norm: 0.1501)
99   PowerCo_Brown → E      : 26.73903 MWh (norm: 0.0994)

```

100 SteelCo_BF → S : 1.18525 ton (norm: 0.0637)
 101 SteelCo_EF → S : 1.18536 ton (norm: 0.0637)
 102 ManuCo_Advanced → C : 5.44797 unit (norm: 0.6441)
 103 ManuCo_Basic → C : 2.82178 unit (norm: 0.3336)
 104 ServiceCo → T : 86.37383 serv-h (norm: 0.5662)

105
 106 FD TOTALS BY PRODUCT:

Product	Total	Target Range	Status
E	67.09752 MWh	50.0- 80.0 MWh	✓
S	2.37061 ton	2.0- 5.0 ton	✓
C	8.26975 unit	8.0- 10.0 unit	✓
T	86.37383 serv-h	80.0-120.0 serv-h	✓

114
 115 NON-FD IMBALANCES ANALYSIS

116
 117 Avg |normalized imbalance|: 0.00000218 ✓
 118 Total products: 5

119
 120 Imbalances:

121 ManuCo_Advanced → Scrap : -0.00003 ton (norm: -0.0000)
 122 MiningCo → IronOre : 0.00007 ton (norm: 0.0000)
 123 ManuCo_Basic → Scrap : 0.00003 ton (norm: 0.0000)
 124 ManuCo_Advanced → C_Unfinished: 0.00000 unit (norm: 0.0000)
 125 ManuCo_Basic → C_Unfinished: 0.00000 unit (norm: 0.0000)

127
 128 ORIGIN FRACTIONS

129
 130
 131 E → ManuCo_Advanced:

132 PowerCo_Green : 0.9645
 133 PowerCo_Brown : 0.0355

134
 135 E → ManuCo_Basic:

136 PowerCo_Brown : 0.8520
 137 PowerCo_Green : 0.1480

138
 139 E → MiningCo:

140 PowerCo_Brown : 1.0000
 141 PowerCo_Green : 0.0000

142
 143 E → ServiceCo:

144 PowerCo_Green : 0.7038
 145 PowerCo_Brown : 0.2962

146
 147 E → SteelCo_BF:

148 PowerCo_Brown : 0.6160
 149 PowerCo_Green : 0.3840

150
 151 E → SteelCo_EF:

152 PowerCo_Green : 0.7928
 153 PowerCo_Brown : 0.2072

154
 155 S → ManuCo_Advanced:

156 SteelCo_EF : 0.6497
 157 SteelCo_BF : 0.3503

158
 159 S → ManuCo_Basic:

160 SteelCo_BF : 1.0000
 161 SteelCo_EF : 0.0000

162
 163 Scrap → SteelCo_EF:

164 ManuCo_Advanced : 0.5607
 165 ManuCo_Basic : 0.4393

167
 168 FINAL DEMAND ORIGIN FRACTIONS (from FD values)

169
 170
 171 C → FINAL (Total: 8.270 unit):

172 ManuCo_Advanced : 0.6588
 173 ManuCo_Basic : 0.3412

174
 175 E → FINAL (Total: 67.098 MWh):

176 PowerCo_Brown : 0.3985
 177 PowerCo_Green : 0.6015

178
 179 S → FINAL (Total: 2.371 ton):

180 SteelCo_BF : 0.5000

```

181 SteelCo_EF      : 0.5000
182
183 =====
184 DIRECT CO2 EMISSIONS BY ENTITY
185 =====
186
187 Total direct CO2 emissions: 116,987.3 kg CO2-eq
188
189 Breakdown by entity:
190 Entity          kg CO2-eq      Share
191 -----
192 PowerCo_Brown   82,893.6      70.9%
193 SteelCo_BF      22,941.0      19.6%
194 PowerCo_Green   8,266.0       7.1%
195 MiningCo        1,764.7       1.5%
196 ManuCo_Basic    436.4         0.4%
197 SteelCo_EF      342.5         0.3%
198 ServiceCo       179.5         0.2%
199 ManuCo_Advanced 163.5         0.1%
200
201 =====
202 RAW DATA FOR REPRODUCIBILITY
203 =====
204
205 PROCESS SCALES:
206 ManuCo_Advanced - Finishing_Advanced : scale 6.5402 → 5.45 unit C
207 ManuCo_Advanced - Stamping_Advanced : scale 6.5402 → 6.54 unit C_Unfinished + 4.58 ton
208 ManuCo_Advanced - Scrap
209 ManuCo_Basic - Finishing_Basic : scale 3.7624 → 3.01 unit C
210 ManuCo_Basic - Stamping_Basic : scale 3.7624 → 3.76 unit C_Unfinished + 3.76 ton
211 ManuCo_Basic - Scrap
212 MiningCo - IronOreExtraction : scale 17.6470 → 17.65 ton IronOre
213 PowerCo_Brown - PowerGen_Brown : scale 103.6170 → 103.62 MWh E
214 PowerCo_Green - PowerGen_Green : scale 165.3195 → 165.32 MWh E
215 ServiceCo - ServiceProvision : scale 152.5493 → 152.55 serv-h T
216 SteelCo_BF - SteelMake_BF : scale 11.7646 → 11.76 ton S
217 SteelCo_EF - SteelMake_EF : scale 6.8510 → 6.85 ton S
218
219 PHYSICAL TRANSACTIONS:
220 ManuCo_Advanced → FINAL : C = 5.447971 unit
221 ManuCo_Advanced → SteelCo_EF : Scrap = 4.033144 ton
222 ManuCo_Basic → FINAL : C = 2.821781 unit
223 ManuCo_Basic → SteelCo_EF : Scrap = 3.160368 ton
224 MiningCo → SteelCo_BF : IronOre = 17.646952 ton
225 PowerCo_Brown → FINAL : E = 26.739029 MWh
226 PowerCo_Brown → ManuCo_Advanced : E = 2.319583 MWh
227 PowerCo_Brown → ManuCo_Basic : E = 51.291180 MWh
228 PowerCo_Brown → MiningCo : E = 1.764701 MWh
229 PowerCo_Brown → ServiceCo : E = 12.199677 MWh
230 PowerCo_Brown → SteelCo_BF : E = 3.623665 MWh
231 PowerCo_Brown → SteelCo_EF : E = 5.679158 MWh
232 PowerCo_Green → FINAL : E = 40.358490 MWh
233 PowerCo_Green → ManuCo_Advanced : E = 63.082226 MWh
234 PowerCo_Green → ManuCo_Basic : E = 8.906818 MWh
235 PowerCo_Green → MiningCo : E = 0.000001 MWh
236 PowerCo_Green → ServiceCo : E = 28.988634 MWh
237 PowerCo_Green → SteelCo_BF : E = 2.258652 MWh
238 PowerCo_Green → SteelCo_EF : E = 21.724701 MWh
239 ServiceCo → FINAL : T = 86.373825 serv-h
240 ServiceCo → ManuCo_Advanced : T = 43.601208 serv-h
241 ServiceCo → ManuCo_Basic : T = 22.574249 serv-h
242 SteelCo_BF → FINAL : S = 1.185247 ton
243 SteelCo_BF → ManuCo_Advanced : S = 3.054638 ton
244 SteelCo_BF → ManuCo_Basic : S = 7.524749 ton
245 SteelCo_EF → FINAL : S = 1.185359 ton
246 SteelCo_EF → ManuCo_Advanced : S = 5.665604 ton
247 SteelCo_EF → ManuCo_Basic : S = 0.000001 ton
248
249 INDIVIDUAL TRANSACTION PRICES:
250 ManuCo_Advanced → FINAL : C = 23249.72 EUR/unit
251 ManuCo_Advanced → SteelCo_EF : Scrap = 269.06 EUR/ton
252 ManuCo_Basic → FINAL : C = 15981.31 EUR/unit
253 ManuCo_Basic → SteelCo_EF : Scrap = 274.76 EUR/ton
254 MiningCo → SteelCo_BF : IronOre = 131.73 EUR/ton
255 PowerCo_Brown → FINAL : E = 71.18 EUR/MWh
256 PowerCo_Brown → ManuCo_Advanced : E = 71.38 EUR/MWh
257 PowerCo_Brown → ManuCo_Basic : E = 74.41 EUR/MWh
258 PowerCo_Brown → MiningCo : E = 75.27 EUR/MWh
259 PowerCo_Brown → ServiceCo : E = 72.81 EUR/MWh
260 PowerCo_Brown → SteelCo_BF : E = 74.95 EUR/MWh
261 PowerCo_Brown → SteelCo_EF : E = 73.57 EUR/MWh

```

```

260 PowerCo_Green → FINAL : E = 83.36 EUR/MWh
261 PowerCo_Green → ManuCo_Advanced : E = 82.82 EUR/MWh
262 PowerCo_Green → ManuCo_Basic : E = 84.68 EUR/MWh
263 PowerCo_Green → MiningCo : E = 86.68 EUR/MWh
264 PowerCo_Green → ServiceCo : E = 83.40 EUR/MWh
265 PowerCo_Green → SteelCo_BF : E = 86.21 EUR/MWh
266 PowerCo_Green → SteelCo_EF : E = 84.95 EUR/MWh
267 ServiceCo → FINAL : T = 148.39 EUR/serv-h
268 ServiceCo → ManuCo_Advanced : T = 144.15 EUR/serv-h
269 ServiceCo → ManuCo_Basic : T = 145.61 EUR/serv-h
270 SteelCo_BF → FINAL : S = 825.95 EUR/ton
271 SteelCo_BF → ManuCo_Advanced : S = 812.53 EUR/ton
272 SteelCo_BF → ManuCo_Basic : S = 831.28 EUR/ton
273 SteelCo_EF → FINAL : S = 915.24 EUR/ton
274 SteelCo_EF → ManuCo_Advanced : S = 922.29 EUR/ton
275 SteelCo_EF → ManuCo_Basic : S = 912.46 EUR/ton
276
277 OPERATING COSTS PER PROCESS:
278 Finishing_Advanced : 9614.32 EUR/scale = 11541.80 EUR/unit
279 Finishing_Basic : 5122.10 EUR/scale = 6402.63 EUR/unit
280 IronOreExtraction : 107.09 EUR/scale = 107.09 EUR/ton
281 PowerGen_Brown : 69.27 EUR/scale = 69.27 EUR/MWh
282 PowerGen_Green : 78.05 EUR/scale = 78.05 EUR/MWh
283 ServiceProvision : 104.44 EUR/scale = 104.44 EUR/serv-h
284 Stamping_Advanced : 4807.17 EUR/scale = 4807.17 EUR/unit
285 Stamping_Basic : 2561.05 EUR/scale = 2561.05 EUR/unit
286 SteelMake_BF : 496.36 EUR/scale = 496.36 EUR/ton
287 SteelMake_EF : 200.39 EUR/scale = 200.39 EUR/ton

```

Listing 2: Optimized state: data/state.txt

F.3 Optimization details

This appendix details the balancing procedure summarized in Section 6.2; the complete implementation is available in the supplementary code.

F.3.1 Parameterization

Process scales are parameterized in log space to ensure positivity. Sourcing fractions are derived from unnormalized scores via softmax, guaranteeing $\sum_i \Lambda_{ijk} = 1$ for each buyer-product pair. Transaction-specific prices and per-process labor costs are similarly parameterized in log space. Child processes with fixed technological coupling (e.g. `Finishing` following `Stamping`) have their scales deterministically linked to their parent.

F.3.2 Optimization

All parameters are optimized simultaneously via gradient descent (Adam optimizer, initial learning rate 0.1 reduced by factor 0.5 on plateau, 10 000 iterations). The loss function combines weighted penalties for: material balance of non-final-demand products (strict); final demand quantities within target ranges; deviations from sourcing preferences; entity markups within target ranges; price consistency across similar transactions (including expected green premiums where applicable) and within plausible absolute ranges. Weights prioritize physical feasibility over softer economic constraints (see `LOSS_WEIGHTS` in F.1).

The optimization converges to a fully balanced state satisfying all constraints, as pictured in F.2.