Beyond Sector Averages: Disaggregated Input-Output Methods for Individual-Level Environmental Accounting

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Abstract

Modern societies operate as a resource-hungry, pollutant-emitting *superorganism* - one that is rapidly depleting *Environmental Assets* (EAs), our collective term for both resources and pollution sinks. Given the finite stocks of EAs, our current economic system is inherently unsustainable - facing inevitable contraction or collapse. To navigate the coming *great simplification* with minimal disruption, humanity must urgently develop systems for coordinated EA governance. Earlier work suggests that such self-organization is fundamentally impossible without **imputing EA usage to individuals** through comprehensive footprinting systems - a prerequisite that simultaneously enables the majority of viable EA governance policy pathways. This exploratory note lays the groundwork for such a method.

We begin by examining environmental input-output methods and observe that – as later understood – existing approaches are mathematically equivalent in their ability to allocate EA responsibility. Working within this unified framework, we build on Charpentier's *Impact Inheritance* (IH) variant and complete it by introducing a missing *supply-side* counterpart (IH-Leontief) to the standard *demand-side* formulation (IH-Ghosh). This enables consistent EA imputation to both key economic drivers: consumption expenditures and productive activities (labor and capital).

Second, we develop methodology to extend these models from sector-level to **fully disaggregated transaction-scale accounting** - enabling EA footprint calculation for individual products and specific value-added activities. Unlike aggregated sector data, these granular measurements directly map to individual human actors, and yield actionable and empirically verifiable results. The mathematical formulation preserves the elegant structure of aggregated IH models while operating on atomic economic units - maintaining all products and entities as distinct inputs *and* outputs. Implementation requires two data components: *process tables* (entity-specific production recipes) and *transaction records* (buyer-supplier-product flows) - with practical acquisition challenges analyzed. To our knowledge, these advances establish the first viable framework for individual EA accounting - theoretically sound and practically achievable through systemic upgrades - laying the foundation for transformative environmental governance.

Context and disclaimer
This note began as a personal 'self-note' (written mostly in May 2025) — an attempt to teach myself the intricacies of environmental input-output analysis by writing everything down. As I dove into the topic, I found that while the high-level, sector-scale concepts are well-established in academia, the practical path to achieving true, transaction-scale (3D) accounting remained unclear. This note is the product of that exploration. It is part literature review, part synthesis of existing ideas in my own words, and part new proposal.
The primary novel contribution is found in Section 3: Disaggregated Input-Output analysis for fine-grained EA use imputation, which details a methodology for operating at a fully disaggregated 'micro-scale'. **A secondary contribution** is the unified treatment of bidirectional (demand/supply-side) responsibility, both in its classical form and its extension to this new micro-scale context. If you are very familiar with IOA, you jump directly to Section 3. I intend to develop the core contributions into a formal paper in the near future. The preceding sections largely constitute a re-analysis and synthesis of the existing field, occasionally supplemented with insights that may be novel.

Updates since initial writing
Subsequent to writing this note, I have come to understand that all existing two-dimensional approaches for allocating environmental responsibility are mathematically equivalent.

- the classical Environmentally-Extended (EE) description,
- the Impact Inheritance (IH) description, and
- the Product Carbon Content (PCC) description [57] - though this last one is not yet incorporated into the present text.

This equivalence was not initially apparent to me. A separate paper that formally relates these methods and demonstrates their equivalence is currently in preparation.

Notation Disclaimer This note intentionally departs from the standard notation common in input-output analysis (e.g., $\mathbf{Z}$, $\mathbf{x}$, $\mathbf{v}$, $\mathbf{f}$), which becomes cumbersome and unclear when extended to the higher-dimensional arrays required for fine-grained, transaction-scale modeling. Instead, this work adopts Einstein summation notation and tensor conventions, which are better suited for this generalization and are standard in fields dealing with multi-dimensional data structures. A detailed explanation of the notation used can be found in Appendix A.

1 Context and related work: we need *individual-level environmental accouting*

Before diving into how to impute resources and pollutants at the individual level, let us expose some context and related works about why we would like to do this, and what precisely is this imputation objective.

1.1 Environmental assets

Human activity From nature's perspective, modern societies function as a mindless superorganism [43] feeding on environmental resources and emitting pollutants. Energy availability dictates the extraction of all other resources.

Environmental assets (EAs) Resources are finite, and so is the environment's capacity to absorb pollutants while sustaining conditions beneficial to humans, animals, and plants. This dual constraint leads us to unify resources and (the absence of) pollutants under the concept of *environmental assets (EAs)*, where 'using' an EA means either depleting a resource or releasing a pollutant. Crucially, most

EA are non-substitutable: their depletion cannot be offset by substituting another EA [13, 5, 11, 15]. If any non-substitutable EAs is exhausted, human civilization as we know it will cease to function.

We consume EAs too fast Many environmental assets are being plundered at unsustainable rates - resources that took millions of years to form vanish in decades, while pollutants persist for millennia or forever. This imbalance - between extraction and regeneration - fuels ecological overshoot [6, 16, 7, 49], cascading into biodiversity collapse, supply chain failures, climate change, and natural disasters. Ultimately, it destabilizes both ecosystems and societies, driving human suffering through food shortages, forced migration, economic collapse, economic inequatity, and violent conflict over dwindling resources.

1.2 Responding to the metacrisis requires individual-level environmental accounting

This set of crises is sometimes grouped under the term *metacrisis* [1] - further referenced in Appendix F. Metacrisis mitigation necessitates order-of-magnitude reductions in EA consumption [34, 61] - requiring unprecedented lifestyle transformations [47, 54]. But these societal shifts remain virtually unattainable without individual-level monitoring systems.

Governing the commons Faced with finite EAs and a superorganism that 'grows but does not voluntarily shrink' [43], we must redesign economic systems to enable equitable sharing - *before* decarbonization and sustainable resource use becomes possible. Elinor Ostrom's seminal work [9] identified eight principles for sustaining shared resources, from Swiss grazing commons to Philippine irrigation systems. These principles - essentially *making the superorganism mindful* - include clear boundaries, collective rule-making, and graduated sanctions. Her framework now underpins polycentric climate governance [31, 39] and modern commons-based solutions [42].

Monitoring EA consumption Ostrom's fourth principle [9] is unambiguous: *every* observed long-lasting commons had a system to monitor individual EA use ('who takes how much?'). While hypothetical exceptions might exist, assuming they do would be reckless. This makes EA monitoring *a non-negotiable foundation* for any organisational solution. This principle is necessary but insufficient; the other seven (boundary-setting, conflict resolution, etc.) remain equally vital - and arguably also much more challenging. However, *monitoring stands out as a pragmatic starting point*: its technical focus requires less immediate global coordination, yet its implementation could catalyze action on the remaining principles. A functioning EA accounting system wouldn't just track harm - it would also encourage communitites and institutions to confront the need for deeper systemic change.

Practical Policy pathways for EA governance To start respecting safe and just Earth system boundaries [62] ideally requires fundamendally reorienting modern society's relationship with the environment [33, 35]. With voluntary transition failing to materialize, operatinalizing effective commons governance requires implementing two complementary policy approaches:
two complementary policy classes must coexist:

- **Reforming EA use constraints**. Current constraint systems such as the EU Emissions Trading Scheme (ETS) [78] represent progress but remain flawed: Their *partial coverage* exempts high-impact sectors (e.g. agriculture, aviation) while loopholes - like the over-allocation of free allowances [20] and carbon leakage rules [73] - dilute effectiveness even in covered sectors. Additionally, such flat-rate approaches face three core flaws: regressive impacts (disproportionate burden on low-income households, diminished effectiveness (largest consumers are price-insensitive), and unaddressed legacy (no mechanism to address wealth accumulated from past overuse). *Progressive individual EA taxes* [64, 52] could resolve these issues. The policy landscape offers a continuum of solutions, bounded by two key approaches: at the one end, the least progressive

[1] refering to the fact that components create risks greater than the sum of their parts, because they reinforce each other. As a result, they cannot be addressed in isolation but rather require a profound, systemic rethink of coordination ad value systems.

option is existing flat-rate taxes, at the other end, the most progressive option is equal per-capita entitlements: fixed, non-tradable EA allowances ensuring basic equity. A range of hybrid approaches offer intermediate options. See also remark 4 of 3.5.2 for a practical remark on implemementing these solutions.

- **Targeted investments and incentives**. A second policy class involves directing public funds - via grants, bonds, or central bank operations - into companies, projects, or financial instruments that reduce EA use. Crucially, these investments require rigorous evaluation using *EA reduction efficiency*: the amount of EA savings achieved per monetary unit spent. Yet, despite public perception that robust 'green finance' tools exist, current systems fail to deliver. National governments lack standardized metrics to assess EA efficiency in spending programs such as green stimulus, infrastructure [45]. Even progressive central banks pursuing climate-aligned policies (e.g., ECB, PBoC) lack EA efficiency tools - highlighting systemic gaps in green central banking frameworks [48, 82, 81]. So-called 'sustainable' finance often subsidizes business-as-usual under an environmental veneer [50, 51]. This measurement vacuum reduces public investment and green finance to performative policymaking, prioritizing optics over verifiable EA reductions.

Informational empowerment A growing cohort of environmentally conscious consumers actively seeks transparency about the environmental impact of their purchases, demanding data to align spending with sustainability values [30].

In short, whether for governance (managing the commons), policy implementation (progressive taxation, hybrid allowances, or EA-weighted investments), or purely informational purposes, the same foundational capability is essential: precise, granular EA consumption footprinting at individual, corporate, and institutional levels. This note focuses on enabling that capability.

1.3 Core Terminology and Economic Structure

We begin by establishing key terminology for the analytical framework that will allow individual-level environmental accounting.

System boundary While most input-output analyses of environmental asset (EA) use operate at national scales, requiring more complex treatment of trade flows via Multi-Regional Input-Output (MRIO) models [23, 29], this work adopts a simplified closed-economy assumption, considering either the global economy as a single closed system, or an economically isolated region where imports/exports are negligible. This simplification allows clearer exposition of core methodological innovations while preserving compatibility with MRIO extensions.

Entities This economy is made up of *entities*. An entity is an economic unit that is legally accountable, periodically auditable, and operationally distinct. This includes public and private corporations, non-profit organizations, and government agencies.

Transactions Economic transactions represent bilateral exchanges where a seller provides a specified quantity of goods or services to a buyer in return for monetary payment. Each transaction is characterized by three fundamental attributes: the physical quantity of goods/services transferred, the monetary value exchanged, and the associated environmental footprint. These transactions form the atomic units for economic analysis and environmental impact accounting in our framework.

Final demand Economic output serves either intermediate consumption (goods/services used as production inputs) or final demand (end-use consumption). Final demand comprises household consumption, government expenditure, and gross capital formation - including both fixed investment and inventory changes. For simplicity, we treat final demand as an aggregated sector in this note.

Value added Production groups generate additional monetary value through their economic activities, measured as the difference between output value and intermediate input costs. This value added represents the net economic contribution at each production stage and comprises four components: employee compensation, gross operating surplus, net taxes on production, and capital depreciation. For simplicity, we also maintain value added as an aggregated measure.

Direct EA consumption Each EA is consumed from the environment by entities who operate *source processes*, each quantifiable through monitoring systems[2]. For example, CO_2 emissions primarily originate from fossil fuel combustion, land-use changes, and industrial processes. Each entity operating such source process is assigned a *direct EA consumption* value based on monitored quantities. The precise definition of source processes involves methodological choices (e.g., initially attributing 'source' fossil fuel emissions to refiners - who then pass them on - versus directly to end-users), with the GHG Protocol [83] offering one established framework. Crucially, such definitions affect intermediate accounting but do not alter the final EA allocation results provided the chosen nomenclature remains consistent economy-wide.

1.4 EA consumption footprinting

Two ways of viewing environmental responsibility Economic entities make and receive investments, run their activities, and make products and services that are sold to other entities and final customers. In doing so, they consume environmental assets. There are two ways of considering and modeling *EA consumption responsibility* in such an economy [18, 19]:

- *Demand-side responsibility - or expenditure responsibility*: consumers, through their expenditures, drive production across economic sectors. Firms organize their operations to meet final demand. Under this approach, we attribute EA consumption responsibility to final demand spenders: household consumers, investors, and the government.

- *Supply-side responsibility - or income responsibility*[3]: workers and capital owners, through their contributions to production, enable the output of economic entities. Economic entities operate by mobilizing labor, capital, and other value-added inputs. Under this approach, we attribute EA consumption responsibility to income earners: employees (wages), investors (profits), and other value-added claimants.

Demand-side and supply-side approaches are complementary The *demand-side (pull)* approach links emissions to consumption, highlighting how households, investors, and governments drive production through spending. This can inform policies like eco-labeling, EA taxes on goods, or shifts in consumption patterns. The *supply-side (push)* approach ties emissions to income generation (wages, profits), exposing how labor and capital enable production - supporting policies such as green job subsidies, capital investment regulations, or value-added-based EA pricing. While demand-side measures target the endpoint of economic chains, supply-side interventions reshape their foundations. Together, they address the full cycle: demand-side reduces emissions by steering consumption, while supply-side incentivizes cleaner production. Policymakers need both levers to avoid burden-shifting and achieve systemic change.

Individual attribution of environmental footprints Final demand and value added can both be mapped to individuals. Final demand includes household consumption (direct to persons), government spending (allocable per capita), and investments (attributable to owners). Value added splits into wages (workers), profits (dividends to shareholders), and taxes (citizens). By tracing final demand and value added to *human actors* — not abstract entities — we can pinpoint true environmental responsibility and enable targeted policies, from progressive environmental-adjusted taxes to equitable transition plans.

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[2] Note that effective EA constraints would require robust auditing mechanisms to complement monitoring.

[3] traditionally termed *producer responsibility*, we find this label misleading as it seems to suggest allocation to production entities rather than value-added beneficiaries (e.g., workers, capital owners). We instead use *income responsibility* or *supply-side responsibility*.

Working with multiple EAs Ultimately, we should strive to map the consumption of EAs to individual human actors for as many EAs as possible—beginning with key ones like carbon emissions, microplastics, endocrine disruptors, rare metals, sand, water use, and biodiversity impacts. For the sake of simplicity, the rest of this note focuses on footprinting a single EA. Methodologically, footprinting multiple EAs can be done by simply applying the single-EA approach multiple times. *Note: Some EAs (e.g., water) require adapted accounting as their use transforms rather than depletes the resource.*

Vocabulary As established, our goal is to *impute* - that is, to *assign measurable responsibility for* - EA usage to individual human actors. Throughout this note, we employ interchangeable terms for this concept, including EA consumption/use, footprinting/imputation/allocation of EAs, individual environmental footprint. All refer to the quantitative attribution of environmental impacts to individuals. This approach aligns with 'cradle-to-gate' attribution, focusing solely on production-phase EA use without life cycle analysis.

1.5 Input-output analysis vs the environmental ledger: two imputation methods

We examine a closed economy over a **given time interval**, during which the direct EA use of all entities can be quantified. Our objective is to impute direct EA use to either final demand (following the consumer responsibility approach) or value added (following the income responsibility approach). Two main classes of methods are conceivable:

- *Environmental Input Output (EIO)* analysis is a set of *global, ex-post* methods that uses mathematical allocation rules to split direct EA use based on consolidated transaction data between entities. This well-established framework, the focus of our work, is detailed and extended in following sections.
- *Environmental-ledger (e-ledger)*-based allocation is a *local, real-time* approach in which an additional quantity, the e-ledger value, is exchanged alongside monetary payments and the traded goods or services in each transaction. This emerging approach is discussed in more depth and contrasted with IOA methods in Appendix D.

> **Disclaimer** On the term 'e-ledger': In this note, I use 'e-ledger' to refer to the general concept of setting environmental liability at the time of transaction. This is a broad methodological definition. It is not entirely clear to me how other initiatives—such as the E-Ledger Institute (which initially used a different name [74]) — precisely perform their calculations, and my use of the term may not match their specific technical meaning. A more precise term may be needed in the future to avoid confusion.

Readiness levels These two methods differ significantly in their current levels of maturity and applicability. Input-Output (IO) Analysis is well-established and widely applied at the scale of aggregated sectors and product groups. However, it does not yet exist - either in theory or practice - at a finer resolution. Developing such a framework for granular EA allocation constitutes the key contribution of this note. E-ledger allocation, by definition, operates at a fine-grained transactional level. While theoretically promising, it remains largely untested in practice, with only a few isolated experimental implementations to date[68, 88, 86]. Also, a supply-side version of it has yet to be formalized.

This note: IOA focus We advocate for prioritizing IOA-derived methods as a more immediately realistic approach, building on an established theoretical foundation. As detailed later in Section 3, adapting IOA to fine-scale resolution requires enhanced transactional data collection and characterization of entities' internal processes - both achievable through extensions of existing Enterprise Resource Planning (ERP) systems, which already centralize financial and operational data. This contrasts sharply with e-ledger systems, which would require entirely new infrastructure for pre-transaction e-liability negotiation - effectively creating a 'second currency' for environmental accounting. We view this as fundamentally more complex to implement in the short term. Notably, these approaches need not be mutually exclusive. As discussed in Appendix D, IOA and e-ledger systems could eventually operate synergistically, with IOA providing macro-scale validation for micro-level e-ledger implementations.

2 Input-Output foundations: from Canonical to Impact Inheritance models

Input-output analysis. Wassily Leontief's foundational input-output (IO) framework[1, 8, 4] quantifies inter-industry dependencies by relating core economic variables - production, value added, final demand, and optionally environmental variables (resource use or emissions, and their allocations). The framework has two canonical forms, determined by analytical direction: the demand-driven *Leontief* model and the supply-driven *Ghosh* model. Each can be applied in three key variants - yielding a total of six models:

- **Canonical (C) variant** (C-Leontief [1] and C-Ghosh [3]): models *production responses* to exogenous changes in final demand (Leontief) or value added (Ghosh).
- **Environmentally-extended (EE) variant** (EE-Leontief [4] and EE-Ghosh [18]): models sector-level *EA use responses* to exogenous variable changes. EA use is imputed to aggregate final demand (Leontief) or value added (Ghosh) by default; only when these are decomposed (e.g., into household consumption, government spending, exports - or product categories by coordinate projections) can EA use be allocated to specific components. Unlike IH models — which track inherited EA impacts across intermediate transactions - EE models attribute impacts *without explicit tracing* of transaction-level inheritance.
- **Impact inheritance (IH) variant** (IH-Leontief - introduced in this note - and IH-Ghosh [53]): traces how environmental impacts are *embedded and inherited* across all transactions, assigning EA values to each intermediate flow in the EA table. Unlike EE models, IH requires no final demand / value added decomposition, directly computing inherited EA burdens for all flows - and by design, imputing total EA use to final demand / value added coordinate components.

Table 1 summarizes and contrasts the two environmental variants: environmentally extended (EE) and impact inheritance (IH). Crucially, due to their distinct analytical principles, EE and IH models of the same form (Leontief or Ghosh) operate in opposing directions. The EE-Ghosh model treats primary inputs (value added) as exogenous, imputing EA use to decomposed value added components, whereas the IH-Ghosh model takes EA use as an exogenous input and propagates it to outputs (final demand). Conversely, the EE-Leontief model treats outputs (final demand) as exogenous, imputing EA use to decomposed final demand components, while the IH-Leontief model takes EA use as an exogenous output and propagates it to inputs (value added).

> A separate paper that formally relates these methods - including the Product Carbon Content formulation [57] - not yet incorporated in this note - and demonstrates their equivalence is currently in preparation.

Advantages of Impact Inheritance (IH) Models. While EE models dominate the literature, IH variants remain understudied. Charpentier recently proposed the downstream-oriented IH-Ghosh version (*impact inheritance* is his naming) in [53] ; to our knowledge, this work presents the first formulation of the upstream-oriented IH-Leontief variant. We argue that IH variants are superior for fine-scale environmental impact imputation at the individual level: they accomplish imputation in a single computational step without requiring decomposition - a critical advantage when handling billions of products and entities. Moreover, by assigning embedded environmental asset (EA) values to every transaction, IH models provide inherent analytical flexibility across scales. While IH models do aggregate EA origin information during imputation, this is inconsequential: constraint formulation does not require origin information. This section formally presents both IH variants (Leontief and Ghosh) using unified notation for EA use imputation, creating a structured foundation for the fine-scale methodological developments in Section 3. For broader background on canonical (C) and environmentally-extended (EE) model variants, see Appendix C.

IH-Ghosh (consumer responsibility)

imputes *total* EA use (as an *input*) to *final demand*, *directly*

- works on embedded footprints Ψ, with *neutral* entities:

$$\tilde{\Psi}_{IJ} = \begin{pmatrix} \psi & \psi_f \\ \psi_d & 0 \end{pmatrix}$$

- imputes proportionally to sales (physical or monetary):

$$B = \hat{\phi}_I^{-1}(\phi|\phi_f) = \hat{\psi}_I^{-1}(\psi|\psi_f) \stackrel{\text{def}}{=\!=} (b|b_f)$$

Imputation process

$$\psi_I^\top = \psi_d (I - b)^{-1}$$

$$\psi = b \odot \psi_I$$

$$\psi_f^{IH} = b_f \odot \psi_I$$

Remark: By decomposing $\psi_d = \sum_j \psi_d^{(j)}$ into *per-producer projections*, it is also possible to obtain final demand EA responsibilities $\psi^{(j)}$ and $\psi_f^{(j)}$ detailed per producer origin j.

IH-Leontief (income responsibility)

imputes *total* EA use (as an *output*) to *value added*, *directly*

- works on embedded footprints Ψ, with *neutral* entities:

$$\tilde{\Psi}_{IJ} = \begin{pmatrix} \psi & \psi_d \\ \psi_w & 0 \end{pmatrix}$$

- imputes proportionally to purchases:

$$A = \begin{pmatrix} \chi \\ \chi_w \end{pmatrix} \hat{\chi}_I^{-1} = \begin{pmatrix} \psi \\ \psi_w \end{pmatrix} \hat{\psi}_I^{-1} \stackrel{\text{def}}{=\!=} \begin{pmatrix} a \\ a_w \end{pmatrix}$$

Imputation process

$$\psi_I = (I - a)^{-1} \psi_d$$

$$\psi = a \odot \psi_I^\top$$

$$\psi_w^{IH} = a_w \odot \psi_I^\top$$

Remark: By decomposing $\psi_d = \sum_i \psi_d^{(i)}$ into *per-producer projections*, it is also possible to obtain value added responsibilities $\psi^{(i)}$ and $\psi_w^{(i)}$ detailed per producer origin i.

EE-Ghosh (income responsibility)

imputes *detailed* EA use to *value added*, by *decomposition*

-
  - EA use $\phi_J = \phi_I^\top$ is proportional to production: $\phi_J = \gamma_J \odot \chi_J$
  - impute assuming fixed sales structure B for Ghosh decomposition: $\chi_I^\top = \chi_w (I - b)^{-1}$
- works on X or Φ, no embedded footprints Ψ

Imputation process:

1. decompose value added into *per-producer projection components*: $\chi_w = \sum_k \chi_w^{(k)}$
2. for each component (k), *detailed* (per producer) EA use responsibility is $\phi_J^{(k)} = \gamma_J \odot \chi_w^{(k)} (I - b)^{-1}$
3. sum over producers j to get *total* EA imputation to value added providers:

$$\psi_w^{EE} = \begin{pmatrix} \sum_j \phi_J^{(1)} & \dots & \sum_j \phi_J^{(n)} \end{pmatrix}$$

EE-Leontief (consumer responsibility)

imputes *detailed* EA use to *final demand*, by *decomposition*

-
  - EA use ϕ_I is proportional to production: $\phi_I = \gamma_I \odot \chi_I$
  - impute assuming fixed technology A for Leontief decomposition: $\chi_I^\top = (I - a)^{-1} \chi_f$
- works on X, no embedded footprints Ψ

Imputation process:

1. decompose final demand into *per-product projection components*: $\chi_f = \sum_k \chi_f^{(k)}$
2. for each component (k), *detailed* (per producer) EA use responsibility is $\phi_I^{(k)} = \gamma_I \odot (I - a)^{-1} \chi_f^{(k)}$
3. sum over producers i to get *total* EA imputation to final demand products:

$$\psi_f^{EE} = \begin{pmatrix} \sum_i \phi_I^{(1)} \\ \dots \\ \sum_i \phi_I^{(n)} \end{pmatrix}$$

Figure 1: Summary of the four EA imputation methods: (IH/EE)-(Ghosh/Leontief). Notation is detailed in Appendix A.

2.1 Fundamentals

Data aggregation and the *single-activity* hypothesis Conventional input-output (IO) presented below - including both the IH model (below) and C/EE models (Appendix C) - operate at the level of *aggregated groups* (or sectors/industries), typically defined by industrial classification (e.g. ISIC/NACE codes). They rely on symmetric IO tables that map transactional flows between sectors ('who buys what from whom'). For the Leontief and Ghosh inverses to retain theoretical validity and economic interpretability, these tables must satisfy the *single-activity hypothesis* (or *pure industries* assumption), enforcing a strict one-to-one correspondence: each industry must produce only one product, and each product must originate from only one industry. In reality, however, most firms engage in multi-product production, violating this requirement. To bridge this gap, statistical transformations impose simplifying (and often unrealistic) assumptions about production technologies or sales structures (Appendix B). The resulting IO tables artificially enforce homogeneity by either *aggregating* each industry's outputs into a single composite product, or *disaggregating* industries into product-specific sub-sectors. Both approaches introduce re-aggregation errors when real-world production structures deviate from these assumptions. While this section adheres to this classical single-activity hypothesis, Section 3 presents our core contribution: a generalized IO methodology that eliminates this constraint, operating at the level of individual entities and products to avoid re-aggregation bias and enable precise modeling of multi-activity entities.

Exchange tables Φ (physical-units) and X (monetary units) Let $\mathbb{E}$ denote the set of all groups and $\mathbb{P}$ the set of all products. We analyze economic exchanges between (and within) groups over a *given time period*. Each group engages in three fundamental types of exchanges: (1) input transactions - obtaining physical input goods or services from other groups (including intra-group flows of products/services and value-added inputs like labor) against money, (2) output transactions - supplying the good or service it produces to intermediate or final demand consumers against money, and (3) EA exchanges with the environment.

These transactions (excluding EA exchanges (3)) are described in Tables Φ and X - shown in Figure 2. We now detail Φ: its core 2×2 submatrix ϕ (small letter) captures inter-group transactions exclusively. Key vectors include: the final demand vector $\phi_f = \phi[:, f]$ and the transposed value-added vector $\phi_w = \phi[w, :]^\top$. Each value ϕ_{ij} represents the *physical quantity of product i purchased by group j*, product-specific units of input i (e.g., steres for wood, liters for fuel). Intra-group transactions may be non-zero ($\phi_{ii} \neq 0$) when groups comprise multiple interacting entities. By convention, final demand consumers neither utilize value-added services nor directly consume EAs - any such EA use is preallocated for simplicity. Table X follows the same logic.

(Physical units)		**Buying group (j)**		**Final demand**
		1	2	
Selling group (i)	1	ϕ_{11}	ϕ_{12}	ϕ_{1f}
	2	ϕ_{21}	ϕ_{22}	ϕ_{2f}
Value added		ϕ_{w1}	ϕ_{w2}	0

(Monetary units)		**Buying group (j)**		**Final demand**
		1	2	
Selling group (i)	1	χ_{11}	χ_{12}	χ_{1f}
	2	χ_{21}	χ_{22}	χ_{2f}
Value added		χ_{w1}	χ_{w2}	0

Figure 2: Exchange tables Φ, in physical units of inputs (i), and X, in monetary units, between groups of a closed economy.

Imputation tables Ψ (EA units) EA flows admit two interpretations [4], which generates two possible versions of an EA imputation table Ψ:

- As inputs (environment providing resources to entities), direct EA use is assigned to groups as *buyers* - and can then be mapped to final consumers (output) who drive demand - yielding the *demand-side EA imputation table* 3a.
- As outputs (environment absorbing emissions/waste), direct EA use comes from groups as *sellers* - and can then be attributed to value-creating producers (input) who enable production - generating the *supply-side EA imputation table* 3b.

These tables are the EA-unit analogs of exchange tables. They *associate an embedded footprint ψ_{ij} - in EA units - to each transaction T_{ij}*. All values in imputation tables Ψ are initially unknown except entries corresponding to direct EA use (marked in red). The core task of IH models is to algorithmically determine these unknown Ψ values - completing the EA liability imputation across the economic network.

Structure and conventions for Ψ

- *Demand-side version* 3a. Value ψ_{ij} quantifies the embedded footprint in sales of group i to group j. The final demand vector is $\psi_f = \Psi[:, f]$. The transposed direct EA use vector $\psi_d = \Psi[d,:]^\top = \phi_d$ represents the known EA use values to be imputed to final demand. By design, the value-added row is null, as demand-side EA allocation focuses exclusively on final-demand products rather than value-added components like employee salaries.
- *Supply-side version*. Table 3b follows similar logic. In this case, the final-demand column is null by design, as supply-side EA allocation focuses exclusively on value-added income rather than final-demand products.

To minimize notational complexity, we default to implicit variant notation (both are denoted Ψ), context will disambiguate - when explicit notation is required we will write respectively Ψ^{FD} and Ψ^{VA}.

Reduced tables $\tilde{\Psi}$ As a summary, each transaction $T_{ij} = (\phi_{ij}, \chi_{ij}, \psi_{ij})$, involves three quantities: a good or service ϕ_{ij}(physical units), money χ_{ij}, and EA liability ψ_{ij} - the latter is determined *ex-post*. As previously introduced, the value-added row of demand-side Ψ is null - and unnecessary for EA imputation to final demand. Likewise, the final-demand column of supply-side Ψ is also null - and unnecessary for EA imputation to value added. We omit them in the following. Henceforth, the term "Ψ" refers implicitly to this reduced form, with the direction (demand or supply-side) determining which dimension is truncated. This simplifies notation without loss of generality.

Marginal sums We define marginal sums on Φ and Ψ:

- *Total outputs* (physical or monetary units) are $\phi_I = \sum_j \Phi_{ij}$ or $\chi_I = \sum_j X_{ij}$, and *total inputs* (monetary units only) as $\chi_J = \sum_i X_{ij}$ (total physical-unit outputs are intentionally undefined due to unit heterogeneity across rows).
- *Total imputation outputs* (EA units) are $\Psi_I = \sum_j \Psi_{ij}$ and *total imputation inputs* (EA units) are $\Psi_J = \sum_i \Psi_{ij}$.

Again, in order to avoid overburdening notation, we do not explicitly discriminate demand-side and supply-side notation. Context will disambiguate.

[4]There interpretations are 'analytical': even an EA traditionally considered a waste can be viewed as a resource in this sense - and vice versa: for example, CO_2 emitted as an output pollutes the atmosphere, while a CO_2-free atmosphere acts as a resource (input) that is depleted when CO_2 is released. The classification of EAs as inputs or outputs thus depends on the imputation framework: we can either attribute resource *extraction* to final demand or assign waste *emissions* to production sectors.

(Demand-side responsibility)		Buying group (j)		Final demand
		1	2	
Selling group (i)	1	ψ_{11}	ψ_{12}	ψ_{1f}
	2	ψ_{21}	ψ_{22}	ψ_{2f}
Value added		0	0	0
Environment		$\psi_{d1} = \phi_{d1}$	$\psi_{d2} = \phi_{d2}$	0

(a) Demand-side imputation table Ψ in EA units, showing consumer responsibility for environmental inputs in a closed economy. The table allocates EA *resource flows* (from environment to production entities) to final demand consumption. Blue shading indicates exogenous variables; orange denotes model-determined values.

(Supply-side responsibility)		Buying group (j)		Final demand	Environment
		1	2		
Selling group (i)	1	ψ_{11}	ψ_{12}	0	$\psi_{1d} = \phi_{1d}$
	2	ψ_{21}	ψ_{22}	0	$\psi_{2d} = \phi_{2d}$
Value added		ψ_{w1}	ψ_{w2}	0	0

(b) Supply-side imputation table Ψ in EA units, showing income responsibility for environmental outputs in a closed economy. The table allocates EA *waste flows* (from production entities to environment) to value-added income recipients. Blue shading indicates exogenous variables; orange denotes model-determined values.

Figure 3: Imputation tables Ψ - in their two variants for IH-Ghosh 3a and IH-Leontief 3b models - are the EA-unit analogs of exchange tables. All values are initially unknown except entries corresponding to direct EA use (red with blue shading), which inherit their value directly from Φ. IH-IOA models determine the unknowns - effectively imputing EA use liability across the economic network.

2.2 IH-Ghosh model

First formulated by Charpentier [53], the IH-Ghosh model solves demand-side EA use responsibility. Given the exchange table Φ [5]. (Figure 2), it determines all values in the imputation table Ψ (Table 3a) - effectively redistributing total direct EA use by all groups onto final demand products. To achieve this, it allocates supplier footprints to customers proportionally to their share in the supplier's total sales.

Neutrality of entities and imputation goal Unlike Φ, Ψ uses consistent units across inputs and outputs, enabling direct comparison of row-wise and column-wise sums.

- *Entity neutrality.* Entities neither accumulate nor decumulate EA units[6] - total inputs and total ouputs are equal.

$$\sum_j \psi + \psi_f = (\sum_i \psi + \psi_d)^\top \tag{1}$$

$$\psi_I = \psi_J^\top \tag{2}$$

- *Complete EA imputation.* Total direct EA use must equal total final demand footprints - which is what we want: impute EA use to final demand products.

$$\sum_j \psi_{dj} = \sum_i \psi_{if} \stackrel{\text{def}}{=\!=} \psi_d^{tot} \tag{3}$$

Together, these two equalities imply that row and column sums of Ψ coincide:

$$\Psi_I = \Psi_J^\top \stackrel{\text{def}}{=\!=} \begin{pmatrix} \psi_I \\ \psi_d^{tot} \end{pmatrix}$$

where scalar ψ_d^{tot} is the total EA use allocated.

Compact form of the imputation problem Combining the structure of Tables Φ and Ψ with the imputation objective and entity neutrality, we obtain the following *compact formulation*:

IH-Ghosh model data:

$$\Phi_{IJ} = \begin{array}{c|cc} \text{i} \setminus \text{j} & |\mathbb{E}| & 1 \\ \hline |\mathbb{E}| & \phi & \phi_f \end{array} \xrightarrow[\sum_j]{} (\phi_I) \tag{4}$$

$$\Psi_{IJ} = \begin{array}{c|cc} \text{i} \setminus \text{j} & |\mathbb{E}| & 1 \\ \hline |\mathbb{E}| & \psi & \psi_f \\ 1 & \psi_d & 0 \end{array} \xrightarrow[\sum_j]{} \begin{pmatrix} \psi_I \\ \psi_d^{tot} \end{pmatrix} \tag{5}$$

$$\Big\downarrow \textstyle\sum_i$$

$$\left(\psi_I^\top \;\middle|\; \psi_d^{tot} \right)$$

Main allocation principle We allocate input EA use to output customers *proportionally to physical quantities transacted*: if entity A buys twice as much of a product as entity B from the same supplier, it inherits twice the EA liability. This implies equality between the row-normalized versions of Φ and Ψ (see Appendix A for notation), formalized as:

$$B = \hat{\phi}_I^{-1} \left(\phi \;\middle|\; \phi_f \right) = \hat{\psi}_I^{-1} \left(\psi \;\middle|\; \psi_f \right) \stackrel{\text{def}}{=\!=} \left(b \;\middle|\; b_f \right) \tag{6}$$

Here, B represents each producer i's *output distribution* - the relative proportions of its physical production sold to each customer j. *Dimensions*: ϕ, ψ and b are $n \times n$ inter-group matrices and ϕ_f, ψ_f and b_f are $n \times 1$ final-demand column vectors.

[5] Physical units ensure physically realistic footprints when producers practice price discrimination (i.e., sell identical products at different unit prices to different buyers). When all transactions occur at uniform unit prices, monetary units yield equivalent results.

[6] Analogous to monetary capital in value added, EA capital could be modeled via dedicated rows. Here we assume neutrality for simplicity.

Allocation Recall that in Φ, all components are known, whereas in Ψ only $\psi_d = \phi_d$ is known initially. Our goal is to compute ψ and ψ_f. A 'vertical' inventory on Ψ gives:

$$\sum_i \psi + \psi_d = \psi_I^\top$$

or, expressing the central part ψ as a function of b (which is known) and the sum vector ψ_I:

$$\psi_I^\top b + \psi_d = \psi_I^\top$$

and if $(I - b)$ is invertible, we get sum vector ψ_I as:

$$\psi_I^\top = \psi_d (I - b)^{-1} \tag{7}$$

Final demand imputation values follow as:

$$\psi_f = b_f \odot \psi_I \tag{8}$$

and **the problem is solved**: we have imputed direct EA use to final demand, while simultaneously determining all intergroup EA exchanges:

$$\psi = b \odot \psi_I \tag{9}$$

Another way to calculate ψ_I - Power series approximation For large-scale economies (where b is high-dimensional), direct inversion of $I - b$ may be computationally intractable. Instead, we leverage the power series representation[7]

$$(I - b)^{-1} = I + \sum_{k=1}^{\infty} b^k, \tag{10}$$

yielding the EA allocation:

$$\psi_I^\top = \psi_d \left(I + \sum_{k=1}^{\infty} b^k \right). \tag{11}$$

Each term $\psi_d b^k$ represents k-tier upstream EA *involvement* of entities (in the sense of remark 1 of 2.4) allowing practical truncation when $||b^k||$ becomes negligible. See interpretation in C.2.1 for more details.

Convergence Equations 7 and 11 are valid only if $(I - b)$ is invertible, or equivalently if the series converges. Appendix C.2.2 provides some context on when this is true and what to do if it is not.

Remark on Convergence While input-output matrices at the sector level are typically well-conditioned, this property is not guaranteed to extend to the micro-scale. The presence of 'hub firms', 'pure intermediate firms', and other specific network structures in a disaggregated economy is likely to introduce significant convergence challenges. Consequently, the naive Neumann series (power iteration) approach may prove insufficient. Solving these large-scale, fine-grained linear systems will likely require modern iterative solvers and sophisticated preconditioners. We therefore identify the development of robust numerical methods for this problem as a critical area for future research.

Example Appendix C.4 shows two simple, toy examples that help understand this method of EA use allocation.

[7] to show this, stop sum at order k=K, multiply both sides by $I - b$, a telescopic sum appears and terms cancel each other out by pair, $I + b^K$ remains, by definition of a productive economy (see paragraph on convergence) $\lim_{k\to\infty} b^k = 0$, and with $k \to \infty$ we get the result.

2.3 IH-Leontief model

As an equivalent of the IH-Ghosh model for the other analytical direction, we propose the IH-Leontief model, which solves supply-side EA use imputation. Given the exchange table X, it determines all values in the imputation table Ψ (Table 3b) - effectively redistributing total direct EA use by all groups onto value-added recipients (e.g., labor, capital) - To achieve this, it allocates each producer's footprint to its suppliers proportionally to their monetary share in the producer's total purchases. A monetary units exchange table is essential here[8].

Allocation goal and neutrality of entities Similar to the demand-side version higher-up, we now formulate allocation objectives for the supply-side version.

- *Entity neutrality.* Entities neither accumulate nor decumulate EA units[9] - total inputs and total ouputs are equal.

$$\sum_j \psi + \psi_d = \left(\sum_i \psi + \psi_w\right)^\top \tag{12}$$

$$\psi_I = \psi_J^\top \tag{13}$$

- *Complete EA imputation.* Total direct EA use must equal value-added footprints - which is what we want: impute EA use to value added income.

$$\sum_i \psi_{id} = \sum_j \psi_{wj} \overset{\text{def}}{=\!=} \psi_d^{tot} \tag{14}$$

Together, these two equalities imply that row and column sums of Ψ coincide:

$$\Psi_I = \Psi_J^\top \overset{\text{def}}{=\!=} \begin{pmatrix} \psi_I \\ \psi_d^{tot} \end{pmatrix}$$

where scalar ψ_d^{tot} is the total EA use allocated.

Compact form of the imputation problem Combining the structure of Tables X and Ψ with the imputation objective and entity neutrality, we obtain the following *compact formulation*:

IH-Leontief model data:

$$\tilde{X}_{IJ} = \begin{array}{cc} \text{i} \setminus \text{j} & |\mathbb{E}| \\ \begin{array}{c} |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c} \chi \\ \hline \chi_w \end{array}\right) \end{array} \tag{15}$$

$$\Big\downarrow \Sigma_i$$

$$\left(\chi_I^\top\right)$$

$$\tilde{\Psi}_{IJ} = \begin{array}{cc} \text{i} \setminus \text{j} & \begin{array}{cc} |\mathbb{E}| & 1 \end{array} \\ \begin{array}{c} |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c|c} \psi & \psi_d \\ \hline \psi_w & 0 \end{array}\right) \end{array} \xrightarrow[\Sigma_j]{} \begin{pmatrix} \psi_I \\ \psi_d^{tot} \end{pmatrix} \tag{16}$$

$$\Big\downarrow \Sigma_i$$

$$\left(\psi_I^\top \,\middle|\, \psi_d^{tot}\right)$$

[8] Monetary units are essential here because (1) value-added allocation (to wages, dividends, etc.) inherently operates in monetary terms. (2) Column-wise proportionality demands uniform units across inputs - a requirement physical units cannot satisfy when aggregating different input factors such as labor and raw materials.

[9] Analogous to monetary capital in value added, EA capital could be modeled via dedicated rows. Here we assume neutrality for simplicity.

Main allocation principle We allocate output EA use to input suppliers *proportionally to monetary transaction values*: if entity A's sales to entity C are twice entity B's (monetarily), A bears twice the EA liability. This implies equality between the column-normalized versions of X and Ψ (see Appendix A for notation), formalized as:

$$A = \begin{pmatrix} \chi \\ \chi_w \end{pmatrix} \hat{\chi}_I^{-1} = \begin{pmatrix} \psi \\ \psi_w \end{pmatrix} \hat{\psi}_I^{-1} \overset{\text{def}}{=\!=} \begin{pmatrix} a \\ a_w \end{pmatrix} \tag{17}$$

Here, a represents each producer j's *input distribution* - the relative proportions of its monetary spendings affected to each supplier i. In Leontief models, it is called the *technological coefficients* matrix. *Dimensions*: χ, ψ and b are $n \times n$ inter-group matrices and χ_w, ψ_w and b_w are $1 \times n$ value-added row vectors.

Allocation Recall that in X, all components are known, whereas in Ψ only $\psi_d = \chi_d$ is known initially. Our goal is to compute ψ and ψ_w. A 'horizontal' inventory on Ψ gives:

$$\sum_j \psi + \psi_d = \psi_I$$

or, expressing the central part ψ as a function of a (which is known) and the sum vector ψ_I:

$$a\psi_I + \psi_d = \psi_I$$

and if $(I - a)$ is invertible, we get sum vector ψ_I as:

$$\psi_I = (I - a)^{-1}\psi_d \tag{18}$$

Value added imputation values follow as:

$$\psi_w = a_w \odot \psi_I^\top \tag{19}$$

and **the problem is solved**: we have imputed direct EA use to value added, whilst simultaneously determining all intergroup EA exchanges:

$$\psi = a \odot \psi_I^\top \tag{20}$$

Power series approximation Again, column vector ψ_I can be written as a power series approximation:

$$\psi_I = \left(I + \sum_{k=1}^{\infty} a^k\right)\psi_d \tag{21}$$

Each term $a^k\psi_d$ represents k-tier downstream EA *involvement* of entities (in the sense of remark 1 of 2.4) induced by direct emissions imputed to value added. This allows practical truncation when $||b^k||$ becomes negligible. See interpretation in C.2.1 for more details.

Convergence Again, refer to C.2.1 for the existence of $(I - a)^{-1}$ - or convergence of $\sum_{k=1}^{\infty} a^k$.

2.4 Observations and interpretation

Having computed all values in the demand-side (Table 3a) and supply-side (Table 3b) imputation tables, we highlight key interpretative insights:

1. **(Ghosh or Leontief) Involvement values**. The vector ψ_I does not sum to ψ_d^{tot} or any other directly meaningful aggregate - and nor should it[10]. Rather, each ψ_{I_i} quantifies the EA use *involvement* of group i: the total EA use that 'flows through' its operations (buying inputs, running processes, selling outputs). This metric empowers groups to reduce their involvement through supplier selection, process optimization, and direct EA use reduction.

[10] to see it on a very simple example, imagine a 'fully descending' economy (as in 9) where only the most upstream group has direct EA use Ψ_{d_1} and no entity sells any product to final consumers except the most downstream group.

2. **(Ghosh) Embedded product footprints**. In the Ghosh model, the allocated values ψ_{ij} represent *embedded product footprints* - the EA liability embodied in goods/services traded between i and j. These footprints capture *iteratively cumulated* EA impacts through circular dependencies (e.g., steel producers sell to toolmakers, who supply tools back to steel plants) and multi-stage production chains (e.g., ore → steel → machinery). They do not sum to total EA use, due to recursive allocation in these loops, and double counting of intermediate flows in multi-stage chains.

3. **(Leontief) Embedded revenue footprints**. Similarly, in the Leontief model, values ψ_{ij} represent *embedded revenue footprints* - the EA liability per monetary earnings by i from j. For the same reasons as in the Ghosh model (economic loops and multi-stage chains), these footprints do not sum to total physical EA consumption.

4. **(Ghosh) Product EA intensities**. The embedded footprint ψ_{ij}^{FD} of product i sold to j scales with the traded quantity ϕ_{ij}. This yields the *product EA intensity*:

$$\gamma_i^{FD} = \frac{\psi_{ij}^{FD}}{\phi_{ij}}, \tag{22}$$

an invariant property of product i (independent of j).

5. **(Leontief) Revenue EA intensities**. The embedded revenue footprint ψ_{ij}^{VA} scales with the monetary value χ_{ij} of product i sold to j. This defines the *revenue EA intensity*:

$$\gamma_j^{VA} = \frac{\psi_{ij}^{VA}}{\chi_{ij}}, \tag{23}$$

an invariant property of producer j (independent of i).

6. **Contrast with *direct* production intensities**. EE models presented in C.3 assume fixed *production intensities* γ (EA use per monetary-unit production), defined as:

$$\gamma_i = \frac{\psi_{d_i}}{\chi_{I_i}} \tag{24}$$

Unlike *imputed* product/revenue intensities (which account for upstream/downstream chains), production intensities reflect *direct* EA use only.

7. **Exporting impact factors**. The embedded EA use intensities (γ_i^{FD}, γ_j^{VA}) derived in remarks 4 and 5 - such as GHG emission factors - can be directy transfered to accounting frameworks like the *GHG Protocol* standards [83]. However, crucially, the GHGP's methodology is neither designed for nor effective in calculating fine-grained, actionable footprints; its limitations (and relation to IOA/e-ledger approaches) relating to this application are detailed in Appendix E.

8. **No meaningful group footprint**. The concept of a 'group footprint' is ill-defined in this framework, where EA use is allocated to final demand or value added—not production. What we rigorously quantify is *group involvement*, *final demand footprints*, or *value-added income footprints*. Some studies [58] attempt to force ψ_{I_i} to sum to ψ_d^{tot} through alternative allocations - while compatible with our methodology, such adaptations exceed the scope of this note.

9. **Hybrid responsibility approaches**. Our framework treats consumer (demand-side) and income (supply-side) responsibilities as distinct imputation strategies. While we focus on these separately, we note that the methodology is fundamentally compatible with hybrid responsibility schemes [58, 19, 25] that combine both perspectives through weighted allocation - as long as can be expressed for the IH model variant. Such integrations - while analytically valuable for specific policy contexts - require normative weighting choices that lie beyond our present scope.

10. ***per-direct-EA-user* detailed views**. IH models directly impute *total* EA use (aggregated across all direct users in the chain) to final demand products (or value-added providers), and does so without requiring input decomposition. EE models, by contrast, provide *disaggregated*

EA use (by direct user) for final demand/value added, but require separate computations for each component (e.g., household consumption, exports - or coordinate projections). Notably, IH models can also produce detailed imputations by decomposing the exogenous variable into per-producer components (e.g., $\psi_{d_I} = \sum_i \psi_d^{(i)}$) and applying the model to each component individually.

3 Disaggregated Input-Output analysis for fine-grained EA use imputation

As outlined in Section 2, classical Input-Output Analysis (IOA) and its environmental accounting extensions operate under the assumption of homogeneous sectors and products. These models treat each sector as having a *single activity*: it purchases inputs from suppliers, generates direct environmental asset (EA) use, and produces a single output — which is then distributed to other sectors or final demand. As detailed in Appendix B, real-world input-output (IO) data - represented in supply and use tables - is not inherently homogeneous: economic sectors often produce secondary products alongside their primary output. To enforce homogeneity, statistical agencies aggregate or disaggregate sectors and product groups according to simple hypotheses on production or sales structure (see B), ensuring a diagonal supply table. This process is feasible because, at coarse resolution, most sectors exhibit limited product diversification. By enforcing homogeneity, each sector can be uniquely associated with its primary product, enabling inter-sector relationships to be represented via square matrices. This simplification supports core IOA applications, such as quantifying how production networks adjust to shifts in final demand or attributing direct pollutant emissions to final products.

However, as we shift toward finer-scale input-output accounting - potentially down to individual entities and products - many entities engage in a large number of activities, and many products are outputs of *multiple entities*. Simple hypotheses such as 'fixed technology' or 'fixed sales structure' previously used for homogenizing entities and products become too unrealistic. Consequently, a traditional exchange table (entity-by-entity or product-by-product) fails to capture the full complexity of economic interactions. To address this, we introduce a novel formalism that preserves fully disaggregated relationships between input/output entities and products. This approach enables principled allocation of environmental asset use to final-demand products or valued-added providers in realistic *multiple-activity* scenarios. Conceptually, our method generalizes the 2D IH-Ghosh and IH-Leontief models (Section 2) to fine-grained data.

3.1 Disaggregated IO tables

We now model entities producing *multiple products*. When an entity outputs multiple products, its sales data alone (whether physical or monetary) cannot properly impute incoming EA use to final-demand outputs or value-added inputs. This limitation occurs because sales records don't reveal how input resources are distributed across different production processes within the entity - what we might call its *internal activity*.

Disaggregated Input-Output tables To enable precise EA use allocation in multi-product entities, we extend traditional IO tables by maintaining full product-entity disaggregation - capturing both internal production processes and sales structure of economic entities in a same table representation. We introduce three core disaggregated tables, sharing almost identical structures: Φ (physical input flows, input product units), X (monetary transactions, currency units), and Ψ (EA use flows, EA units). These tables preserve the Section 2 interpretation but now cover all {product, entity} combinations as distinct inputs/outputs, resulting in significantly larger dimensions. As before, EA flows support two interpretations, generating two versions of Ψ via the *impact inheritance* method:

- *Demand-side responsibility*: EA as *input* is imputed to final-demand products
- *Supply-side responsibility*: EA as *output* is imputed to value-added recipients

For notational simplicity, we maintain the implicit convention for Ψ introduced in Section 2, where the context makes clear whether we refer to the demand-side or supply-side versions of these tables.

Figure 4 illustrates Ψ's structure for both interpretations. Since Φ and X almost share Ψ's structure (except they don't have and EA row or column) we show only Ψ. The vertical axis displays all possible {product, entity} input combinations, featuring two special cases: value added as {VA product, VA providers} and the environment as {EA, environment} (demand-side responsibility only). The horizontal axis enumerates all possible {product, entity} output combinations, with two special cases: final demand as {FD product, FD buyers} and the environment as {EA, environment} (suppy-side responsibility only). This framework preserves the core IO function of tracking input-output relationships while achieving full product-entity disaggregation, simultaneously capturing internal production processes and external sales structures. The environmental interactions are explicitely isolated [11].

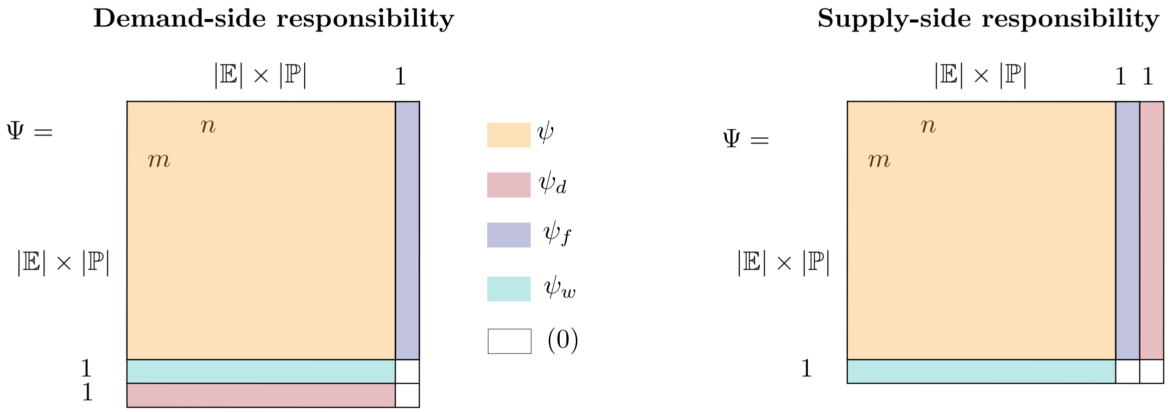


Figure 4: Disaggregated IO table Ψ components: ψ, ψ^f, ψ^w and ψ^d, expressed in physical units of input, shown for both imputation directions. This figure is the disaggregated equivalent of Figure 2. *Left*: Demand-side responsibility treats the environment as an EA resource provider - letting us trace these resources through to final demand products. *Right*: Supply-side responsibility treats the environment as an EA sink - letting us trace these impacts back to value-added income recipients. X and Ψ share identical structures. Best viewed in color. Tables Φ and X follow the same structure, except without the environmental component.

3.2 Observable data: the Process Table and Transaction Table

While the disaggregated IO tables Φ and X cannot be directly observed from economic data, two partially aggregated tables can be constructed from measurable records: the *process table* and the *sales table*. We first introduce these observable structures, then show in the following subsection how to combine them into complete disaggregated tables Φ and X.

Process table Φ^π The *process table* (π for process), denoted $\Phi^\pi = \Phi_{IKL}$, captures the internal production processes of entities. Φ^π has shape $(|\mathbb{P}|+1) \times |\mathbb{P}| \times |\mathbb{E}|$, and each element Φ^π_{ikl} quantifies the amount of input product i required by entity l to produce its output product k. Φ^π is sparse: most entities only use a small subset of all possible inputs and make a small subset of all possible products. Finally, it includes value added as a special input, accounting for the +1 in the first dimension. Φ^π is expressed in physical units of input products (units of i) - and its main part has no monetary equivalent, being fundamentally physical in nature[12] (but its value added part does: χ^π_w). Φ^π can be decomposed as follows:

[11] The {EA, environment} pair appears either as an input row (demand-side responsibility), or as an output column (supply-side responsibility), never combining with other product-entity pairs

[12] In this fine-scale framework, entities may buy a same input product i at different prices depending on supplier. This makes the notion of input product unit price undefined, and makes it impossible to convert the main part ϕ^π into monetary units.

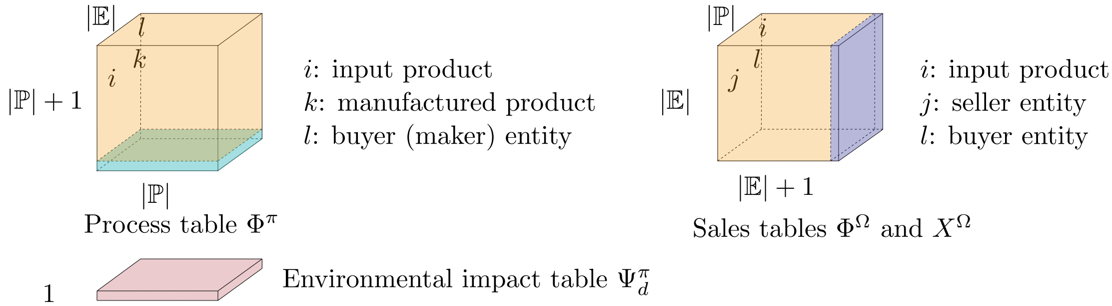


Figure 5: Input data required for fine-scale EA allocation is simply a record of internal processes of entities (Φ^π), a record of transactions (Φ^Ω and X^Ω), and a record of direct environmental impacts (Ψ^π_d). Best viewed in color.

$$\forall k,l \in [\![1,|P|]\!] \times [\![1,|E|]\!], \qquad \Phi^\pi[:,k,l] = \begin{pmatrix} \phi^\pi[:,k,l] \\ \phi^\pi_w[k,l] \end{pmatrix} \tag{25}$$

Section 3.5 discusses practical considerations for obtaining Φ^π.

Sales tables Φ^Ω and X^Ω. The *sales tables* Φ^Ω (physical) and X^Ω (monetary) describe 'who buys what from whom' relationships. Both tables share dimensions $|\mathbb{P}| \times |\mathbb{E}| \times (|\mathbb{E}|+1)$, and each element Φ^Ω_{ijl} (resp. X^Ω_{ijl}) indicates the physical (resp. monetary) quantity of product i that entity l purchases from entity j. Φ^Ω and X^Ω are also sparse: most entities produce few products and sell them to limited buyers. Φ^Ω uses physical units of input products (units of i), while X^Ω uses monetary units. They are related through the price matrix M via $X^\Omega = M \odot \Phi^\Omega$. Finally, they include final demand as a special output, accounting for the +1 in the last dimension. Φ^Ω can be decomposed into its main part and final demand part:

$$\forall i,j \in [\![1,|P|]\!] \times [\![1,|E|]\!], \qquad \Phi^\Omega[i,j,:] = \begin{pmatrix} \phi^\Omega[i,j,:] & \phi^\Omega_f[i,j] \end{pmatrix} \tag{26}$$

And similarly for X^Ω:

$$\forall i,j \in [\![1,|P|]\!] \times [\![1,|E|]\!], \qquad X^\Omega[i,j,:] = \begin{pmatrix} \chi^\Omega[i,j,:] & \chi^\Omega_f[i,j] \end{pmatrix} \tag{27}$$

The sales table is easy to obtain: essentially, it is already in business accounts of entities. Section 3.5 also discusses practical questions on obtaining the data.

Constructing full tables from observable data While we have theoretically defined Φ and X, their practical construction remains to be addressed. These complete disaggregated tables can be derived systematically from the three measurable components Φ^π, Φ^Ω and X^Ω. The following subsections detail this construction for each imputation model.

3.3 Fine-grained IH-Ghosh model

Consistent with the single-activity formulation, the IH-Ghosh model attributes no EA use liability to value added. We therefore remove the value added row from Φ and Ψ. The fine-grained IH-Ghosh model computes final demand EA allocation ψ_f and intermediate flows ψ given direct EA use ψ_d (viewed as input) and production structure Φ. The resulting compact formulation generalizes the single-activity case (Eqs. 4 and 5) to full {product-entity} granularity:

Fine-grained IH-Ghosh model data:

$$\Phi_{MN} = \begin{array}{c} m \setminus n \\ |\mathbb{P}|\times|\mathbb{E}| \end{array} \begin{array}{c} |\mathbb{P}|\times|\mathbb{E}| \quad\quad 1 \\ \left(\begin{array}{c|c} \phi & \phi_f \end{array} \right) \end{array} \xrightarrow[\Sigma_j]{} (\phi_M) \tag{28}$$

$$\Psi_{MN} = \begin{array}{c} m \setminus n \\ |\mathbb{P}|\times|\mathbb{E}| \\ 1 \end{array} \begin{array}{c} |\mathbb{P}|\times|\mathbb{E}| \quad\quad 1 \\ \left(\begin{array}{c|c} \psi & \psi_f \\ \hline \psi_d & 0 \end{array} \right) \end{array} \xrightarrow[\Sigma_j]{} \begin{pmatrix} \psi_M \\ \psi_d^{tot} \end{pmatrix} \tag{29}$$

$$\Big\downarrow \Sigma_i$$

$$\left(\psi_M^\top \,\middle|\, \psi_d^{tot} \right)$$

3.3.1 'Side-vectors' ϕ_d and ϕ_f

The process table's EA component ψ_d^π and transaction table's final demand component ϕ_f^Ω become ψ_d and ϕ_f through dimensional flattening:

$$\psi_d[n] = \psi_d^\pi[k,l] \quad \text{where} \quad n = k|\mathbb{P}| + l \tag{30}$$

$$\phi_f[m] = \phi_f^\Omega[i,j] \quad \text{where} \quad m = i|\mathbb{P}| + j \tag{31}$$

3.3.2 Main part ϕ

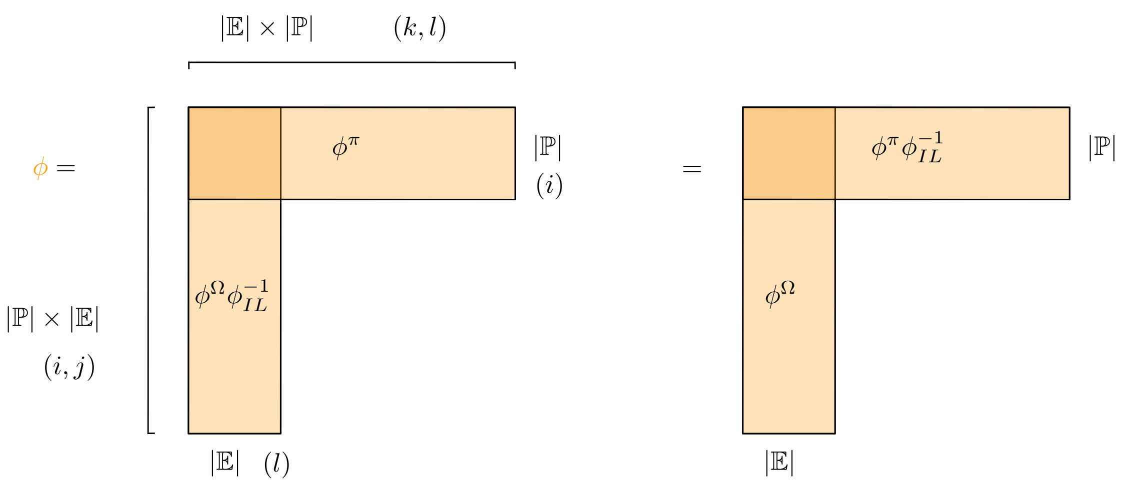


Figure 6: Main part ϕ of $\tilde{\Phi}$ is obtained from the observable tables ϕ^π and ϕ^Ω as $\phi_{IJKL} = \phi^\pi_{IKL} \odot \phi_{IL}^{(-1)} \odot \phi^\Omega_{IJL}$. In practice this can be done either by *expanding* ϕ^π *using* ϕ^Ω*-based coefficients*, of equivalently by *expanding* ϕ^Ω *using* ϕ^π*-based coefficients*.

The 2D table ϕ_{MN} (shape $(|\mathbb{P}| \times |\mathbb{E}|)^2$) from 3.1 equivalently represents a 4D tensor ϕ_{IJKL} (shape $|\mathbb{P}| \times |\mathbb{E}| \times |\mathbb{P}| \times |\mathbb{E}|$) along dimensions I, J, K, L. We construct it from the (main-part) process table $\phi^\pi = \phi_{IKL}$ (from Φ^π), and the (main-part) sales table $\phi^\Omega = \phi_{IJL}$ (from Φ^Ω). As per notations of Appendix A, these are partial sums of ϕ_{IJKL}:

$$\phi_{IKL} = \sum_j \phi_{IJKL} \tag{32}$$

$$\phi_{IJL} = \sum_k \phi_{IJKL} \tag{33}$$

Their consistency requires identical marginal sums over shared dimensions [13]:

$$\phi^{\pi}_{IL} = \sum_k \phi_{IKL} = \sum_j \phi_{IJL} = \phi^{\Omega}_{IL} \tag{34}$$

We now combine ϕ_{IKL} and ϕ_{IKL} to reconstruct the 4D table ϕ_{IJKL} through the tensor operation:

$$\phi_{IJKL} = \phi_{IKL} \odot \phi^{(-1)}_{IL} \odot \phi_{IJL} \tag{35}$$

where $\odot$ denotes the Hadamard (element-wise) product, $\phi^{(-1)}_{IL}$ represents the element-wise inverse of marginal sums, and dimension broadcasting follows Appendix A. The reconstruction admits two equivalent operational interpretations:

ϕ^{π}-Expansion Approach Equation 35 decomposes as:

$$\phi_{IJKL} = \phi^{\pi}_{IKL} \odot \underbrace{(\phi^{(-1)}_{IL} \odot \phi^{\Omega}_{IJL})}_{\text{origin coefficients } \Lambda_{IJL}}$$

Here ϕ^{Ω}_{IJL} is normalized over supplier entities (J), yielding origin coefficients $\Lambda_{IJL} = \phi^{\Omega}_{IJL} \odot \phi^{\Omega}_{IL}{}^{(-1)}$ that satisfy $\Lambda_{IJL} \in [0,1]^{|\mathbb{P}|\times|\mathbb{E}|\times|\mathbb{E}|}$ and $\sum_j \Lambda_{IJL} = 1$ for all (i,l) pairs. These coefficients quantify, for each input product i and buyer entity l, the proportion sourced from supplier j - allowing reconstruction of ϕ_{IJKL} from ϕ^{π}_{IKL}.

ϕ^{Ω}-Expansion Approach The equivalent alternative interpretation (of the exact same equation) is:

$$\phi_{IJKL} = \underbrace{(\phi^{\pi}_{IKL} \odot \phi^{(-1)}_{IL})}_{\text{process coefficients } \Gamma_{IKL}} \odot \phi^{\Omega}_{IJL}$$

Here ϕ^{π}_{IKL} is normalized over output products (K), producing process coefficients $\Gamma_{IKL} = \phi^{\pi}_{IKL} \odot \phi^{\pi}_{IL}{}^{(-1)}$ with $\Gamma_{IKL} \in [0,1]^{|\mathbb{P}|\times|\mathbb{P}|\times|\mathbb{E}|}$ and $\sum_k \Gamma_{IKL} = 1$ for all (i,l) pairs. These coefficients determine, for each input product i and buyer entity l, the proportion used for making output product k - allowing reconstruction of ϕ_{IJKL} from ϕ^{Ω}_{IJL}.

Core Allocation Assumptions ϕ^{π}-expansion assumes that a product's sourcing (dimension J absent from $\phi^p i$) is independent of its production use (dimension K in ϕ^{π}). Equivalently, ϕ^{Ω}-expansion assumes that a product's use (dimension K absent from ϕ^{Ω}) is independent of its supplier (dimension J in ϕ^{Ω}). These assumptions simply mean that products with the same label should be treated identically. If they are not, they should be classified as separate products.

Tensor Structure The 2D representation ϕ_{MN} is simply obtained from ϕ_{IJKL} through standard tensor flattening (trailing dimension first), with index mapping:

$$m = i|\mathbb{P}| + j \qquad \text{(input dimensions } I, J \text{ combined)}$$
$$n = k|\mathbb{P}| + l \qquad \text{(output dimensions } K, L \text{ combined)}$$

This preserves all information while converting the 4D structure $(|\mathbb{P}| \times |\mathbb{E}| \times |\mathbb{P}| \times |\mathbb{E}|)$ to a 2D matrix $(|\mathbb{P}||\mathbb{E}| \times |\mathbb{P}||\mathbb{E}|)$ as required for the IH-Ghosh model (see 3.3.3).

Units and Interpretation Thanks to the normalization operation, ϕ_{IJKL} and ϕ_{MN} are still expressed in *physical units of inputs*, as are ϕ^{π} and ϕ^{Ω}. Each element of ϕ (whether in 4D ϕ_{IJKL} or 2D ϕ_{MN} form) represents the same fundamental relationship: **the physical quantity of input product i from entity j used in producing output product k by entity l.** In other words, the tensor ϕ simultaneously encodes *internal production processes* (input-output transformations within entities) and *external market transactions* (buyer-seller relationships between entities).

[13] Discrepancies indicate data collection errors.

3.3.3 EA Use Imputation

The fine-grained IH-Ghosh model imputes EA flows analogously to the single-activity case (Section 2.2), but now operates on fully disaggregated {product, entity} pairs.

Imputation Constraints The matrix $\tilde{\Psi}$ must satisfy *Input-Output balance*: row sum $\begin{pmatrix} \psi_M \\ \psi_d^{tot} \end{pmatrix}$ and (transposed) column sum are equal. The part on direct EA use ψ_d^{tot} enforces complete allocation of direct EA use to final demand products (as in Eq. 3). The part on intermediates ψ_M enforces perfect pass-through of embedded EA flows at all production stages (as in Eq. 1). The key advancement is that neutrality now applies at the granular {product, entity} level, ensuring all EA liabilities inherited through inputs for a specific product are transferred to that product's buyers upon sale.

Solution The fine-grained solution mirrors the single-activity IH-Ghosh model (Section 2.2), but operates on $|\mathbb{P}| \times |\mathbb{E}|$ dimensional elements rather than $|\mathbb{E}|$ alone. The identical mathematical form yields:

$$\begin{cases} \psi_I^\top = \psi_d(I - b)^{-1} = \psi_d(I + \sum_{k=1}^{\infty} b^k) \\ \psi_f = b_f \odot \psi_I \\ \psi = b \odot \psi_I \end{cases} \tag{36}$$

And the problem is solved.

Producer-Dependent Product EA Intensities All observations from Section 2.4 remain valid. The fine-grained case introduces a key additional insight: for a given product i produced by different entities j and j', the EA use intensities γ_m^{FD} and $\gamma_{m'}^{FD}$ (where $m = i|\mathbb{P}| + j$, $m' = i|\mathbb{P}| + j'$) will generally differ. These intensities are computed from ψ as:

$$\gamma_m^{FD} = \frac{\psi_{mn}^{FD}}{\phi_{mn}} \quad \text{(independent of } n\text{)}$$

This heterogeneity emerges because distinct producers ($j \neq j'$) typically source inputs from different suppliers, and exhibit different production efficiencies - which affects their EA footprint use per unit output.

3.4 Fine-grained IH-Leontief model

Consistent with the single-activity formulation, the IH-Leontief model attributes no EA liability to final demand. We therefore remove the final demand column from X and Ψ. The fine-grained IH-Leontief model computes value-added beneficiary imputation ψ_w and intermediate flows ψ given direct EA use ψ_d (viewed as output) and monetary transaction structure X. The resulting compact formulation generalizes the single-activity case (Eqs. 15 and 16) while operating at full {product-entity} granularity as follows:

Fine-grained IH-Leontief model data:

$$X_{MN} = \begin{array}{c|c} \mathrm{m} \setminus \mathrm{n} & |\mathbb{P}| \times |\mathbb{E}| \\ \begin{array}{r} |\mathbb{P}| \times |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c} \chi \\ \hline \chi_w \end{array}\right) \end{array} \xrightarrow{\Sigma_i} \left(\chi_M^\top\right) \tag{37}$$

$$\Psi_{MN} = \begin{array}{cc} \mathrm{m} \setminus \mathrm{n} & \begin{array}{cc} |\mathbb{P}| \times |\mathbb{E}| & 1 \end{array} \\ \begin{array}{r} |\mathbb{P}| \times |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c|c} \psi & \psi_d \\ \hline \psi_w & 0 \end{array}\right) \end{array} \xrightarrow{\Sigma_j} \left(\begin{array}{c} \psi_M \\ \psi_d^{tot} \end{array}\right), \quad \xrightarrow{\Sigma_i} \left(\psi_M^\top \,\middle|\, \psi_d^{tot}\right) \tag{38}$$

3.4.1 'Side-vectors' ϕ_d and ϕ_w

The process table's EA component ψ_d^Ω and process table's value-added component χ_w^π become ϕ_d and χ_w through dimensional flattening:

$$\psi_d[m] = \phi_d^\Omega[i,j] \quad \text{where} \quad m = i|\mathbb{P}| + j \tag{39}$$

$$\chi_w[n] = \chi_w^\pi[k,l] \quad \text{where} \quad n = k|\mathbb{P}| + l \tag{40}$$

3.4.2 Main part χ

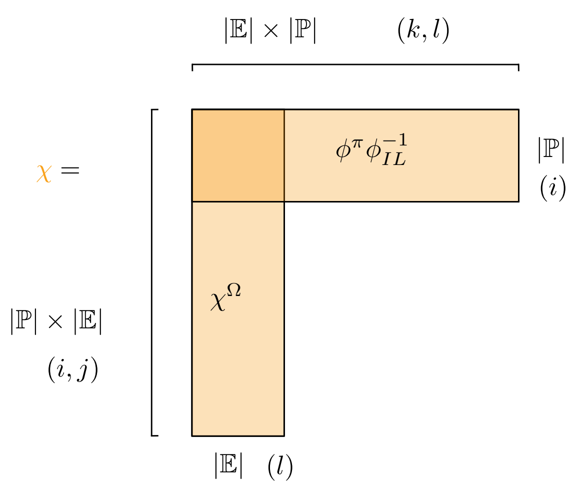


Figure 7: Main part χ of $\tilde{X}$ is obtained from the observable tables χ^Ω and ϕ^π by *expanding* χ^Ω *using* ϕ^π*-based coefficients*: $\chi_{IJKL} = \phi_{IKL}^\pi \odot \phi_{IL}^{(-1)} \odot \chi_{IJL}^\Omega$. Since ϕ^π does not have a monetary-unit equivalent, here there is only one way of obtaining χ, unlike ϕ.

The 2D table χ_{MN} (shape $(|\mathbb{P}| \times |\mathbb{E}|)^2$) from 3.1 equivalently represents a 4D tensor χ_{IJKL} (shape $|\mathbb{P}| \times |\mathbb{E}| \times |\mathbb{P}| \times |\mathbb{E}|$) along dimensions I, J, K, L. We construct it from the (main part) monetary sales table $\chi^\Omega = \chi_{IJL}$ (from X^Ω), and the (main part) physical process table $\phi^\pi = \phi_{IKL}$ (from Φ^π), through the tensor operation:

$$\chi_{IJKL} = \underbrace{\left(\phi_{IKL}^\pi \odot \phi_{IL}^{(-1)}\right)}_{\text{process coefficients } \Gamma_{IKL}} \odot \chi_{IJL}^\Omega \tag{41}$$

where - as in the Ghosh-(ϕ^{Ω}-expansion approach) - ϕ^{π}_{IKL} is normalized over output products (K), producing process coefficients $\Gamma_{IKL} = \phi^{\pi}_{IKL} \odot \phi^{\pi}_{IL}{}^{(-1)}$ with $\Gamma_{IKL} \in [0,1]^{|\mathbb{P}|\times|\mathbb{P}|\times|\mathbb{E}|}$ and $\sum_k \Gamma_{IKL} = 1$ for all (i,l) pairs. These coefficients determine, for each input product i and buyer entity l, the proportion used for making output product k, - allowing reconstruction of ϕ_{IJKL} from χ^{Ω}_{IJL}.

Again, this expansion assumes that a product's use (dimension K absent from χ^{Ω}) is independent of its supplier (dimension J in χ^{Ω}). Flattening of 4D tensor χ_{IJKL} into 2D table χ_{MN} follows identical index mapping as ϕ_{MN}. Thanks to the normalization operation, both are expressed in *monetary units*, as is χ^{Ω}. Each element of χ (whether in 4D χ_{IJKL} or 2D χ_{MN} form) represents the same relationship: **the monetary quantity of input product i from entity j used in producting output product k by entity l.**

3.4.3 EA Use Imputation

The fine-grained IH-Leontief model imputes EA flows analogously to the single-activity case (Section 2.3), but now operates on fully disaggregated {product, entity} pairs.

Imputation Constraints The matrix $\tilde{\Psi}$ must satisfy *Input-Output balance*: column sum $\begin{pmatrix} \psi_M \\ \psi_d^{tot} \end{pmatrix}$ and (transposed) row sum are equal. The part on direct EA use ψ_d^{tot} enforces complete allocation of direct EA use to value added recipients (as in Eq. 14), while the part on intermediates ψ_M enforces perfect pass-through of embedded EA flows at all production stages (as in Eq. 12). Again, the key advancement is upstream neutrality at the granular {product, entity} level, ensuring all EA impacts generated through production are fully allocated to their respective input supply chains.

Solution The fine-grained solution mirrors the single-activity IH-Leontief model (Section 2.2), but now operates on $|\mathbb{P}| \times |\mathbb{E}|$ dimensional elements rather than $|\mathbb{E}|$ alone. The identical mathematical form yields:

$$\begin{cases} \psi_I = (I-a)^{-1}\psi_d = (I + \sum_{k=1}^{\infty} a^k)\psi_d \\ \psi_w = a_w \odot \psi_I^{\top} \\ \psi = a \odot \psi_I^{\top} \end{cases} \tag{42}$$

And the problem is solved.

Value-Added EA Intensity Heterogeneity All observations from Section 2.4 remain applicable. The fine-grained extension also reveals that for a given product k sold to different entities l and l', the value-added EA intensities γ_n^{VA} and $\gamma_{n'}^{FD}$ (where $n = k|\mathbb{P}| + l$, $n' = k|\mathbb{P}| + l'$) will typically differ. These intensities are computed from ψ as:

$$\gamma_n^{VA} = \frac{\psi_{mn}^{VA}}{\chi_{mn}} \quad \text{(independent of } m\text{)}$$

This heterogeneity occurs because different buyers ($l \neq l'$) typically operate distinct production processes and exhibit varying input efficiencies - which shapes their upstream EA footprint per monetary unit received.

3.5 Obtaining data in practice

Practical implementation requires enhanced disclosure of two key datasets: process table Φ^{π} (detailing input-product relationships per production activity) and transaction tables Φ^{Ω} and X^{Ω} (documenting buyer-supplier product flows). Current accounting practices lack this granularity, necessitating coordinated regulatory efforts across jurisdictions [60].

While essential for fine-scale EA allocation, such supply network data would also unlock transformative applications across multiple domains. For climate policy, it enables real-time carbon leakage monitoring and CBAM implementation; for industrial strategy, identifying critical supply chain vulnerabilities;

for macroeconomic modeling, tracking inflation propagation through production networks; and for financial regulation, detecting transfer pricing anomalies and subsidy fraud. These dual environmental and economic benefits could motivate broader compliance, whether through regulation or market incentives.

3.5.1 Obtaining Φ^{Ω} and X^{Ω}

While systematic collection frameworks for transactional data are not yet fully established, the underlying data is inherently available to all economic actors. Each entity already tracks - or can easily record - the precise quantities (ϕ^{Ω}) and monetary values (X^{Ω}) of products sold to specific buyers. Although some businesses may need to enhance their accounting granularity, this requires only marginal adjustments to existing practices, not new technical capabilities. Current initiatives demonstrate the feasibility of such data aggregation [60], including cross-border VAT digitalization efforts [66], mandatory e-invoicing systems [77], and integrated trade data platforms [71]. These real-world implementations confirm that the primary challenge lies in organizational coordination rather than technical feasibility.

3.5.2 Obtaining Φ^{π}

The process table Φ^{π} constitutes the most critical yet challenging component for fine-grained EA allocation. While the precise methodologies for constructing Φ^{π} in practice extend beyond the scope of this note—requiring future work combining measurement science, accounting standards, and policy design—several fundamental considerations emerge when addressing its implementation. These span technical measurement challenges, economic reporting practices, and incentive structures, all of which we examine below to establish a foundation for operationalizing Φ^{π} in real-world systems.

1. **Input Allocation Ambiguity**. Multi-product entities share resources across production processes, creating fundamental allocation challenges when outputs lack natural partitioning rules. A dairy farm cannot objectively split feed consumption and methane emissions between milk and meat, just as a semiconductor fab lacks physical criteria to allocate emissions between chips and silicon dust. The production process itself provides no intrinsic basis for such allocations. Temporal offsets between inputs and outputs can also introduce additional technical complexities.

2. **Allocation Strategy Spectrum**. Given these challenges, practical implementations span a continuum:

    - *Entity-Defined Allocation*: Entities freely declare Φ^{π} without constraints. While simple to implement, this sacrifices physical plausibility and cross-entity comparability, retaining only basic conservation properties.
    - *Standardized Allocation*: Externally defined Φ^{π} tables enforced uniformly across entities. Though ensuring consistency, this oversimplifies real-world process diversity and is only viable for highly standardized industries.
    - *Audited Self-Declaration*: Entities propose Φ^{π} subject to automated validation (including process similarity analytics across peer entities) and spot audits. This balances flexibility with accountability through algorithmic cross-checks and regulatory oversight.

3. **Parallels to Cost Accounting**. Many entities already conduct internal cost accounting, which implicitly constructs Φ^{π} through splitting input expenditures across activities to assess profitability; monitoring cost drivers for strategic resource allocation, informing pricing strategies through activity-based costing. Though currently non-standardized and internal, these practices effectively populate Φ^{π}, suggesting existing workflows could be adapted for EA reporting with minimal added burden.

4. **Progressive Taxation and ϕ^{π} Distortion Incentives**. Progressive environmental taxes (e.g., carbon taxation based on personal EA footprints [64]) create perverse incentives when ϕ^{π} reporting is unconstrained. Entities will strategically optimize both ϕ^{π} allocations and pricing - shifting reported EA use toward products consumed by lower-tax-rate customers while implementing compensatory pricing to maintain market share. For instance, vehicle manufacturers

might disproportionately allocate emissions to economy models (facing lower marginal tax rates) while inflating luxury vehicle prices to compensate. This market driven strategy games the system - distorting estimated footprints to boost competitiveness. This proves the need for hard constraints on ϕ^π to keep progressive environmental taxes honest and effective.

5. **Universal ϕ^π Challenges**. These allocation questions are inherent to *any* fine-grained EA accounting system—not a limitation unique to our method. Whether through IOA extensions (like ours) or e-ledgers, all high-resolution frameworks must resolve input allocation ambiguities, balance flexibility with comparability, and prevent strategic manipulation. This universality underscores the urgency of developing **standardized ϕ^π collection protocols** across methodologies.

In summary, fine-scale EA footprinting non-negotiably requires ϕ^Ω, X^Ω (transaction records) and ϕ^π (process rules). Realizing this demands immediate, coordinated effort to: (1) Establish ϕ^Ω and X^Ω reporting infrastructures, (2) Develop standardized ϕ^π allocation methods through industry-regulator collaboration— with ϕ^π standardization remaining the critical bottleneck.

4 Conclusion

Individual environmental asset footprinting serves as an indispensable yet incomplete tool for metacrisis mitigation. While Ostrom's full framework of commons governance principles - including boundary-setting, participatory rule-making, and sanctions - remains essential, EA monitoring offers a uniquely pragmatic starting point: it is both technically implementable and politically achievable. The strategic path is clear: should Europe, China, and allied economies mandate comprehensive EA accounting, their collective economic influence could drive global adoption. By tethering environmental responsibility to individuals, we expose current inequities in EA use, and create the precise measurement framework needed to design and implement effective policies, transforming monitoring into a catalyst for systemic change.

This note establishes the theoretical foundation for fine-grained environmental impact imputation by extending input-output methods. We build on the demand-side Impact Inheritance (IH-Ghosh) model developed by Charpentier (2022) and introduce its novel supply-side counterpart: the IH-Leontief model, enabling bidirectional impact attribution. Second, we develop a generalized formulation that scales these dual methodologies to fully disaggregated levels, demonstrating how to construct the required high-dimensional tensors from two practical data sources: entity-level process tables (internal production data) and transaction records (sales data). This bridges theoretical IO constructs with implementable accounting systems, enabling precise environmental footprinting at individual entity and product levels.

5 Perspectives

We now stand at the threshold of a transformative era in sustainability governance, where precise individual-level environmental footprinting transitions from theoretical framework to operational reality. While the core mathematical foundations are established, multiple critical engineering challenges remain to be solved to deliver reliable measurement:

- *Industrial-Scale Computation*: Building optimized, preconditioned solvers for ultra-large Ghosh/Leontief systems
- *Partial Participation Frameworks*: Establishing rigorous hybrid approaches that apply exact IH methods to certified supply chain segments, use conservative estimation for uncertified components, and embed automatic participation incentives.
- *Uncertainty propagation*: Incorporate uncertainty in any input data.

- *Process Data Standardization*: Developing standardized and auditable methods for allocating shared inputs across multi-output production systems (this one is not only an engineering challenge)
- *Dynamic Accounting Systems*: Creating continuous protocols that handle EA flows across reporting periods

These measurement advances can unlock key policy implementation pathways, such as gaming-resistant constraints that impose fair, progressive limits on individual environmental asset use, and verification ecosystems (process table safeguards, cross-validated auditing frameworks).

Realizing this vision requires unprecedented collaboration across computational mathematics, industrial engineering, and policy design communities - with the shared goal of turning environmental accountability from a theoretical ideal into working global infrastructure.

A Glossary of notation

Table 1 summarizes the key notations used throughout this note. Note that the components of exchange tables—such as value added, final demand, and EA use—are excluded here and discussed in detail in the main text. By convention, capital letters denote primary matrices and tensors (e.g., Φ, X, Ψ), where the same symbol may represent different versions of a table (e.g., IH-Ghosh, IH-Leontief, or their disaggregated forms) depending on context; explicit distinctions are omitted for brevity. Lowercase letters represent components of these primary tables, which may take the form of vectors (e.g., ϕ_d, ψ_f), matrices (e.g., ϕ, a, ϕ_d^π), or tensors (e.g., ϕ^Ω, ϕ_{IJKL}). Unless otherwise stated, all vectors are assumed to be column vectors.

Notation	Shape	Meaning
Sets		
$\mathbb{E}$	$\lvert\mathbb{E}\rvert$	Set of economic entities (sectors)
$\mathbb{P}$	$\lvert\mathbb{P}\rvert$	Set of products
General		
$T = T_{IJKL}$	$\lvert\mathbb{I}\rvert \times \lvert\mathbb{J}\rvert \times \lvert\mathbb{K}\rvert \times \lvert\mathbb{L}\rvert$	Generic 4D tensor
$T_{IJK} = \sum_l T$	$\lvert\mathbb{I}\rvert \times \lvert\mathbb{J}\rvert \times \lvert\mathbb{K}\rvert$	Reduction of T by summation over dimensions l and m
$T_{IJ} = \sum_{kl} T$	$\lvert\mathbb{I}\rvert \times \lvert\mathbb{J}\rvert$	Reduction of T by summation over dimensions k, l, and m
IH-Ghosh		
Φ	$(\lvert\mathbb{E}\rvert + 2) \times (\lvert\mathbb{E}\rvert + 1)$	Demand-side, physical-unit exchange table
$\tilde{\Phi}$	$(\lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes value added row)
X	$(\lvert\mathbb{E}\rvert + 2) \times (\lvert\mathbb{E}\rvert + 1)$	Demand-side, monetary-unit exchange table
$\tilde{X}$	$(\lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes value added row)
Ψ	$(\lvert\mathbb{E}\rvert + 2) \times (\lvert\mathbb{E}\rvert + 1)$	Demand-side, EA-unit exchange table
$\tilde{\Psi}$	$(\lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes value added row)
γ^{FD}	$\lvert\mathbb{E}\rvert$	Product EA intensities
IH-Leontief		
Φ	$(\lvert\mathbb{E}\rvert + 1) \times (\lvert\mathbb{E}\rvert + 2)$	Supply-side, physical-unit exchange table
$\tilde{\Phi}$	$(\lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes final demand column)
X	$(\lvert\mathbb{E}\rvert + 1) \times (\lvert\mathbb{E}\rvert + 2)$	Supply-side, monetary-unit exchange table
$\tilde{X}$	$(\lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes final demand column)
Ψ	$(\lvert\mathbb{E}\rvert + 1) \times (\lvert\mathbb{E}\rvert + 2)$	Supply-side, EA-unit exchange table
$\tilde{\Psi}$	$(\lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes final demand column)
γ^{VA}	$\lvert\mathbb{E}\rvert$	Revenue EA intensities
Disaggregated IH-Ghosh and IH-Leontief (common)		
Φ^π	$(\lvert\mathbb{P}\rvert + 2) \times \lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert$	Process table (physical units of input i)
Φ^Ω	$\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert \times (\lvert\mathbb{E}\rvert + 2)$	Transaction table (physical units of input i)
X^Ω	$\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert \times (\lvert\mathbb{E}\rvert + 2)$	Transaction table (monetary units)
Disaggregated IH-Ghosh		
Φ	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 2) \times (\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)$	Demand-side, physical-unit exchange table
$\tilde{\Phi}$	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes value added row)
X	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 2) \times (\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)$	Demand-side, monetary-unit exchange table
$\tilde{X}$	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes value added row)
Ψ	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 2) \times (\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)$	Demand-side, EA-unit exchange table
$\tilde{\Psi}$	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes value added row)
γ^{FD}	$\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert$	Disaggregated product EA intensities
Disaggregated IH-Leontief		
Φ	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1) \times (\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 2)$	Supply-side, physical-unit exchange table
$\tilde{\Phi}$	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes final demand column)
X	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1) \times (\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 2)$	Supply-side, monetary-unit exchange table
$\tilde{X}$	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes final demand column)
Ψ	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1) \times (\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 2)$	Supply-side, EA-unit exchange table
$\tilde{\Psi}$	$(\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert + 1)^2$	Reduced version (excludes final demand column)
γ^{VA}	$\lvert\mathbb{P}\rvert \times \lvert\mathbb{E}\rvert$	Disaggregated revenue EA intensities

Table 1: Notations used in the note

A.1 Algebra Notation

I denotes the identity matrix. The superscript $^\top$ indicates matrix transpose, while $^{-1}$ denotes matrix inverse.

- $(A \mid B)$ represents horizontal concatenation of matrices A and B
- $\begin{pmatrix} A \\ B \end{pmatrix}$ represents vertical concatenation of matrices A and B

For a vector a of length l, $\hat{a} = \mathrm{diag}(a)$ is the diagonal matrix with entries of a on its diagonal.

- Row normalization: For matrix A of shape (l, m), pre-multiplication $\hat{a}A$ scales the rows of A by a (i.e., multiplies each row by the corresponding entry of a)
- Column normalization: For matrix A of shape (m, l), post-multiplication $A\hat{a}$ scales the columns of A by a

These operations are particularly useful for normalizing matrices by their row or column sums (where a is the respective sum vector, and we multiply by $\hat{a}^{-1}$).

Starting from Section 3, we work with *full IO tensors* of dimension 4 and their reductions through summation over different dimensions, producing tensors of decreasing dimensionality (from 4 to 0). For a tensor T, we use explicit dimension notation: T_{IJKL} indicates a tensor of shape $|\mathbb{I}| \times |\mathbb{J}| \times |\mathbb{K}| \times |\mathbb{L}|$. The notation extends to reduced versions through summation:

- $T_{IJK} = \sum_l T_{IJKL}$ (reduction over dimension L)
- $T_{IJ} = \sum_{k,l} T_{IJKL} = \sum_k T_{IJK}$ (reduction over dimensions K and L)

The same notation applies for any combination of remaining dimensions (e.g., T_{JL}). For indexing:

- Lowercase letters denote indices: $T_{IJm} = T_{IJM}[:, :, m]$ is the slice at index m along dimension M
- $T_{IJK}^{(-1)}$ denotes the element-wise inverse: $T_{IJK}^{(-1)}[i, j, k] = 1/T_{IJK}[i, j, k]$

The parentheses in $T^{(-1)}$ distinguish this operation from matrix inversion (relevant for 2D cases).

In tensor operations, $\odot$ denotes the Hadamard product (element-wise multiplication) while standard matrix multiplication uses implicit notation (e.g., AB for matrices A and B). To streamline expressions, we use implicit dimension broadcasting in these operations. For example, $T_{IK} \odot T_K^{(-1)}$ represents column-wise division of T_{IK} by T_K, equivalent to the matrix notation $T_{IK}\hat{T}_K^{-1}$. When tensors appear in sequence without an explicit operator (e.g., AB), we ensure the implied operation is either unambiguous from context, or explicitly annotated with the output shape.

B Standard Practices for IO Table Construction

As noted in 2.1, contemporary Input-Output Analysis (IOA) is typically conducted at a coarse scale, where entities and product groups align with broad sectoral classifications defined by National Accounts standards [26]. Fine-grained IOA has thus far been infeasible due to the absence of both (1) infrastructure capable of recording fine-scale transactions and internal processes and (2) methodological frameworks for performing IOA at high resolution. (This note addresses the latter gap by introducing a method for fine-scale IOA.) To contextualize the process, we examine how Input-Output (IO) tables are constructed at this coarse resolution.

B.1 From Supply-Use Tables to Square Matrices

In practice, IO tables are rarely observed directly; instead, they are analytically derived from *Use Tables (UT)* and *Supply Tables (ST)* (denoted U and V, respectively) [2, 41, 26, 79, 69, 27]. These tables organize raw economic data by product and entity categories, though typically with an asymmetry - the number of product groups often differs from the number of industry groups. The *Use Table* (UT) records the quantities of each product consumed by each industry, while the *Supply Table* (ST) documents the quantities of each product supplied by each industry. This dual structure is particularly well-suited for National Accounts, as it enables cross-verification by calculating aggregate metrics - such as total output or value added - through both supply and use perspectives. The data for these tables are compiled from diverse sources, including administrative records (e.g., tax authorities, customs agencies), statistical surveys, and supplementary industry reports.

As previously noted, Supply-Use Tables (SUTs) are structured as product-by-industry matrices, with the number of products typically exceeding the number of industries. In an idealized scenario where each product is produced by only one industry, constructing an industry-by-industry IO table would be trivial - requiring only the aggregation of product groups. However, real-world economies exhibit *secondary production*, where certain products are supplied by multiple industries, making lossless conversion of SUTs to square IO matrices *mathematically impossible* under general conditions. Yet, square matrices remain non-negotiable for Leontief and Ghosh analysis. The field thus operates on a pragmatic compromise: imposing simplifying assumptions about production technologies or sales structures to force rectangular SUTs into the required square form. Four canonical methods achieve this [41, 79], each combining U, V, and either S (product-by-product) or Z (industry-by-industry) matrices - with hybrid approaches also feasible. We now dissect these approaches. To align with conventional standards [14], we adopt widely-used notation (summarized in Figure 8). For enhanced clarity, we occasionally specify matrix dimensions explicitly - U_{PE} (products $\times$ industries), V_{EP} (industries $\times$ products), Z_{EE} (industry $\times$ industry), and S_{PP} (product $\times$ product).

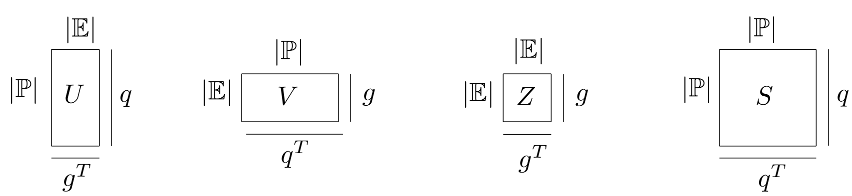


Figure 8: Supply and Use Tables (V and U) and square IO tables Z (industry by industry) and S (product by product), with their dimensions and sum vectors. Here $|\mathbb{E}|$ is the number of *aggregated* entity groups (industries) and $|\mathbb{P}|$ is the number of *aggregated* product groups.

ITA: Industry Technology Assumption The Industry Technology Assumption (ITA) asserts that all products from a given industry share the same production technology. The industry technology matrix $A_{PE} = U\hat{g}^{-1}$ represents product input requirements per unit of industry output. Applying this assumption transforms the supply matrix V_{EP} (recording each industry's product outputs) into

[14] Mapping to our paper's explicit dimensional notation: $U = X_{IL}$, $V = X_{JK}$, $S = X_{IK}$, $Z = X_{JL}$, $g = X_J = X_L$ (industry outputs/inputs), $q = X_I = X_K$ (product outputs/inputs).

the product-by-product IO matrix:

$$S_{PP} = A_{PE}V_{EP} = U\hat{g}^{-1}V, \tag{43}$$

This formulation preserves industry-level production homogeneity while aggregating across product-specific variations.

PTA: Product Technology Assumption The Product Technology Assumption (PTA) postulates that each product has a unique production technology invariant across industries. The product technology matrix $A_{PP} = S\hat{q}^{-1}$ gives input requirements per unit of product output. Enforcing this assumption yields:

$$U_{PE} = A_{PP}V_{EP}^{\top} \quad \text{and thus} \quad S = UV^{-\top}\hat{q}, \tag{44}$$

where $V^{-\top} \equiv (V^{\top})^{-1}$. This requires square invertible V ($|\mathbb{P}| = |\mathbb{E}|$), implying each industry must be a pure producer of exactly one primary product. Practical application necessitates (1) data homogenization to create artificial pure industries, and (2) acceptance of potentially uninterpretable negative coefficients in S, revealing the assumption's fundamental tension with real-world production systems.

ISS: Industry Sales Structure The Industry Sales Structure (ISS) assumption posits that each industry maintains fixed proportional sales to other industries, regardless of product composition. The sales structure matrix $B_{EE} = \hat{g}^{-1}Z$ quantifies inter-industry sales proportions per unit of total industry output. Under ISS, product use patterns derive from industry sales structure as:

$$U_{PE} = V_{EP}^{\top}B_{EE} \quad \text{and thus} \quad Z_{EE} = \hat{g}V^{-\top}U, \tag{45}$$

which again demands square invertible V_{EP} ($|\mathbb{P}| = |\mathbb{E}|$). Like PTA, industries must be preprocessed into (approximated) pure producers of single products, potentially generating non-physical negative coefficients in Z_{EE} when real-world data violates the assumption's strict proportionality.

PSS: Product Sales Structure The Product Sales Structure (PSS) assumption requires each product to maintain fixed proportional sales across industries, regardless of producing sector. The sales structure matrix $B_{PE} = \hat{q}^{-1}U$ captures these product-to-industry sales patterns per unit of product output. Under PSS, industry-level transactions emerge as aggregations of product sales:

$$Z_{EE} = V_{EP}B_{PE}$$

$$Z = V\hat{q}^{-1}U$$

The formulation maintains product-wise sales patterns while aggregating to industry transactions.

B.2 Key Takeaways

Four canonical methods (and their hybrid combinations, e.g., Norway's approach [41, 79]) exist to derive square IO tables from rectangular Supply-Use Tables (SUTs). These methods enforce homogeneity assumptions - industries and products are treated as uniform entities, even when underlying data contradicts this simplification. Such compromises are necessary to apply Leontief and Ghosh models, but they come at a cost: realism is sacrificed for mathematical convenience.

This trade-off becomes increasingly untenable at fine scales. Disaggregated systems exhibit complex, entity-specific production and sales behaviors that cannot be captured by simplistic industry - or product-wide assumptions. Our proposed *fine-grained EA imputation* method directly addresses this limitation. While more data-intensive, it delivers actionable, entity-specific accounts rooted in verified operational data rather than statistical approximations, inherently capturing detailed internal processes and sales structures. The methodological shift we advocate is not merely incremental - it *fundamentally rethinks how input-output systems should be constructed in the era of granular, digital traceability.*

C Six Input-Output models

This part is mostly synthesis of well-known ideas in my own words.

While this note focuses on imputing environmental assets use (EA) responsibility to *individuals* using the IH-Ghosh and IH-Leontief models (developed in the main text), these belong to a broader family of IO models. Two better-known variants exist for each approach: the Canonical (C) versions used for economic analysis and the Environmentally-Extended (EE) versions that explicitely incorporate ecological flows. For completeness and to contextualize our IH models within the full IO taxonomy, we describe all variants here. All notations follow Appendix A, and like in the main text, we maintain the *single-activity* assumption where entities and products are homogeneous.

Remark about the equivalence between the different environmental impact calculation approaches:
Subsequent to writing this note, I have come to understand that all existing two-dimensional approaches for allocating environmental responsibility are mathematically equivalent.

- the classical Environmentally-Extended (EE) description,
- the Impact Inheritance (IH) description, and
- the Product Carbon Content (PCC) description [57] - though this last one is not yet incorporated into the present text.

This equivalence was not initially apparent to me. A separate paper that formally relates these methods and demonstrates their equivalence is currently in preparation. Until then, the explanations below are still valid, if slightly incomplete.

C.1 IOA models: fundamentals

Marginal sums. We define marginal sums on monetary exchange table $\bar{X}$:

- *Total outputs* χ_I is the column-vector of production/earnings per production sector - here measured in monetary units:
$$\chi_I = \sum_j \chi + \chi_f \tag{46}$$
- *Total inputs* χ_J is the row-vector of requirements/earnings per production sector - here measured in monetary units:
$$\chi_J = \sum_i \chi + \chi_w \tag{47}$$

Neutrality of entities. By definition, entities are neutral monetarily: they do not accumulate or decumulate monetary capital[15].

$$\chi_J = \chi_I^\top \tag{48}$$

This also implies that summed final demand $\chi_{f_{tot}} = \sum_i \chi_f$ and summed value added $\chi_{w_{tot}} = \sum_j \chi_w$ are equal. Total gross output throughout the economy can be obtained by summing either total outputs χ_I or total inputs χ_J.

Coefficient matrices A and B. To work with χ, it is useful to normalize the input-output table either:

[15] if they do, this profit is included in value added - see definition of value added in 1.3:

- along columns to obtain [16] the *technology matrix* A:

$$A = \begin{pmatrix} \chi \\ \chi_w \end{pmatrix} \hat{\chi}_I^{-1} \stackrel{\text{def}}{=\!=} \begin{pmatrix} a \\ a_w \end{pmatrix} \tag{49}$$

  Matrix a is the called *technology matrix* and its coefficients are refered to as *input or technological coefficients*. Elements a_{ij} of A quantity the proportions of each input i required to make one output unit of j. These input proportions result from technological constraints - hence the name of A - and are generally considered fixed as final demand changes.

- or along rows to obtain [17] the *allocation matrix* B (monetary version [18]):

$$B = \hat{\chi}_I^{-1} \big(\chi \;\big|\; \chi_f\big) \stackrel{\text{def}}{=\!=} \big(b \;\big|\; b_f\big) \tag{50}$$

  Matrix b is called the *allocation matrix* and its coefficients are refered to as *output or sales or allocation coefficients*. Elements b_{ij} of B quantify, for one output unit of i, the proportions sold to each group j. B is valid for a fixed supply chain structure and coefficients are considered fixed as resource availability or supply changes.

Leontief vs Ghosh models. These matrices are used in two classes of models: Leontief models and Ghosh models. Each class has three versions: canonical (C), environmentally-extended (EE), and impact inheritance (IH). This results in six input-output models in total: Leontief[8], EE-Leontief[4], IH-Leontief (first introduced in this note, to our knowledge), Ghosh[3], EE-Ghosh[18], IH-Ghosh [53].

- Leontief models assume that technology matrix A is constant. Their canonical version (C) examines how changes in final demand affect production. Their environmentally-extended (EE) version examines how changes in final demand affect EA use - or equivalently, imputes direct EA use of entities to *components of final demand*. Their impact inheritance (IH) version determines supply-side individual responsibility - i.e. imputes direct EA use to *value added inputs*.

- Ghosh models assume that sales structure B is constant. Their canonical (C) version examines how changes in value added affect production. Their environmentally-extended version examines how changes in value added affect EA use - or equivalenly, imputes direct EA use of entities to *components of value added*. Their impact inheritance (IH) version determines demand-side individual responsiblity - i.e. imputes direct EA use to *final demand outputs*.

The C-Leontief and C-Ghosh models - traditionally used to predict economic adaptation to variable changes - have sparked extensive debate about whether fixed technology (Leontief) or fixed sales structure (Ghosh) assumptions are more realistic in different contexts [12, 14]. However, since we allocate pollution through static physical variables rather than modeling economic adaptation, these discussions become irrelevant for environmental accountability.

Note As already introduced before, there is also the 'Product Carbon Content' model [57], not yet incorporated in this note, and equivalent to the other two environmental models (EE and IH). Thus far it was only formulated in the 'demand-side' direction - though it also has a 'supply-side' version.

[16] See Appendix A for 'hat' notation. In short: $a_{ij} = \chi_{ij}/\Phi_{J_j}$.

[17] See Appendix A for 'hat' notation. In short: $b_{ij} = \chi_{ij}/\Phi_{I_i}$.

[18] It is also possible to calculate the *physical* allocation matrix: $B^\phi = \hat{\phi}_I^{-1}\big(\phi \;\big|\; \phi_f\big)$. In case sellers do not practice price distrimination (they sell to the same unit price to all sellers), then $B = B^\phi$. For large aggregate sectors this is a reasonable approximation. However, in the fine-scale case this is probably unreasonable, which is why we use B^ϕ in the main part for fine-scale imputation.

C.2 Canonical models

C.2.1 C-Leontief model

Compact form. We can write the monetary exchange table in compact form, including marginal sums and the neutrality constraint:

C-Leontief model data:

$$\tilde{X}_{IJ} = \begin{array}{c|cc} \text{i} \setminus \text{j} & |\mathbb{E}| & 1 \\ \hline |\mathbb{E}| & \chi & \chi_f \\ 1 & \chi_w & 0 \end{array} \xrightarrow{\Sigma_j} \begin{pmatrix} \chi_I \\ \chi_d^{tot} \end{pmatrix}, \quad \downarrow \Sigma_i \quad \begin{pmatrix} \chi_I^\top & \chi_d^{tot} \end{pmatrix} \tag{51}$$

The classic Leontief model is a 'demand-driven' input-output model:

- we assume that final demand χ_f is the exogenous variable (blue background) .
- we would like to calculate production χ_I and its components χ and χ_w for any different value of χ_f, assuming that each group has technologically-fixed input proportions.

Fixed technology assumption: : $A = \begin{pmatrix} \chi \\ \chi_w \end{pmatrix} \hat{\chi}_I^{-1}$ is considered constant when χ_f varies. Let us take the example of a window producer group j requiring aluminium from the aluminium producer group i and other products (glass, energy, etc) from other groups. The fact that the ratio χ_{ij}/χ_j of value of required aluminium to value of total requirements is close to constant makes intuitive sense: it reflects the technological fact that on average, windows have a given proportion of aluminium, and unless window technology changes (for example by using another material), this proportion will not change.

Leontief model. Replacing $\chi = a\hat{\chi}_I$ in the horizontal inventory 46, we get:

$$\chi_I = \sum_j a\hat{\chi}_I + \chi_f$$

$$\chi_I = a\chi_I + \chi_f$$

If $I - a$ is invertible we can calculate $L = (I - a)^{-1}$, known as the *Leontief inverse*, and get the following linear model, that makes it possible to calculate the required production to meet a known demand vector, considering that we can leave A fixed:

$$\chi_I = (I - a)^{-1}\chi_f \tag{52}$$

Value added and inter-sector monetary exchange values follow as:

$$\chi_w = a_w \odot \chi_I^\top \tag{53}$$

$$\chi = a \odot \chi_I^\top \tag{54}$$

and **the problem is solved**.

Power series approximation. With a similar reasoning as in Equation **??**, χ_I can also be written as a power series:

$$\chi_I = (I + \sum_{k=1}^{\infty} a^k)\chi_f \tag{55}$$

Interpretation. Each term $a^k \chi_f$ represents k-tier upstream production: to satisfy final demand, each entity has to produce at least its corresponding part in χ_f. However, for each unit of output of each entity j, additional input $a[:, j]$ from other entities is required - therefore for entities to produce χ_f, their suppliers one level upstream have to produce an additional $a\chi_f$ (intermediate-use production). Now, for entities to make this additional production, they also require additional inputs from their suppliers: $a^2\chi_f$. We can continue like this until $||a^k||$ becomes negligible. The same reasoning can be done (with production or EA use - and downstream effects or upstream requirements) for equations 11, 21 and 60.

Convergence. A real-world economy is *(demand-driven) productive* in the sense of Leontief if a is non-negative and there exists some positive vector χ_I so that $(I - a)\chi_I$ is a positive vector. This is famously equivalent to saying that $(I - a)^{-1}$ exists and is non-negative, or that $\lim_{k\to\infty} a^k = 0$. In a real-world IO table, every group most likely has to pay for non-zero value added, in the form of employee wages or taxes. Therefore, columns of a sum to stricly less than one and spectral radius is $\sigma(a) < 1$ and we get $\lim_{k\to\infty} a^k = 0$. A way to understand this is that the economy can generate a 'surplus', meaning that the outputs exceed the inputs required for production, in other words every group is profitable, or every final demand vector can be met. In practice for an observed real-world IO-table, this should always be the case: if a group cannot satisfy final demand, it goes bankrupt and disappears. Note that in some rare cases such as extreme interdependencies when group 1 gets all its input from group 2 and vice versa, or when group 3 is a linear combination of groups 1 and 2, $I - a$ may lose invertibility. But in such cases, it is easy to regain invertibility by simply merging the groups or moving some financial transactions in the IO table to the next or previous time period. In real-world applications, statistical agencies and researchers often do clean and preprocess data in case of poor quality, significant errors, or unusual economic structures to make sure that $I - a$ is invertible.

C.2.2 C-Ghosh model

Compact form. We can write the monetary exchange table in compact form, including marginal sums and the neutrality constraint:

C-Ghosh model data:

$$\tilde{X}_{IJ} = \begin{array}{c c} & \begin{array}{cc} |\mathbb{E}| & 1 \end{array} \\ \begin{array}{c} |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c|c} \chi & \chi_f \\ \hline \chi_w & 0 \end{array}\right) \end{array} \xrightarrow[\sum_j]{} \begin{pmatrix} \chi_I \\ \chi_d^{tot} \end{pmatrix}, \qquad \downarrow \textstyle\sum_i \qquad \left(\chi_I^\top \,\middle|\, \chi_d^{tot}\right) \tag{56}$$

The classic Ghosh model is a 'supply-driven' input-output model:

- we assume that value-added χ_w is the exogenous variable (blue background) (for example wages, taxes, profits).
- we would like to calculate production χ_I and its components χ and χ_f for any different value of χ_w, assuming that the structure of the suppy chain does not change.

Fixed sales structure assumption. $B = \hat{\chi}_I^{-1}\left(\chi \,\middle|\, \chi_f\right)$ is considered constant as χ_w varies. A fixed supply chain structure means that each group distributes its total output to other groups and final demand in fixed proportions regardless of output levels, reflecting some kind of supply-side rigidity due to relationship with buyers or long-term contracts.

Ghosh model. Replacing $\chi = \hat{\chi}_I^\top b$ in the vertical inventory 47, we get:

$$\chi_I^\top = \sum_i \hat{\chi}_I^\top b + \chi_w$$

$$\chi_I^\top = \chi_I^\top b + \chi_w$$

If $I - b$ is invertible we can calculate $G = (I - b)^{-1}$, known as the *Ghosh inverse*, and get the following linear model, that makes it possible to calculate the production value resulting from a change in primary inputs, considering that we can leave B fixed:

$$\chi_I^\top = \chi_w (I - b)^{-1} \tag{57}$$

Final demand and inter-sector monetary exchange values follow as:

$$\chi_f = b_f \odot \chi_I \tag{58}$$

$$\chi = b \odot \chi_I \tag{59}$$

and **the problem is solved**.

Power series approximation. Again, row vector $\chi_I^\top$ can be written as a power series:

$$\chi_I^\top = \chi_w (I + \sum_{k=1}^{\infty} b^k) \tag{60}$$

Each term $\chi_w b^k$ represents k-tier upstream EA contributions, allowing practical truncation when $||b^k||$ becomes negligible. See interpretation in C.2.1 for more details.

Convergence. Using a similar proof as for a *productive* economy in the production approach C.2.1, we can write the same thing for the the pollutant allocation approach with b: an economy is *(supply-driven) productive* if b is non-negative and there exists some positive vector $\chi_I^\top$ so that $\chi_I^\top (I - b)$ is a positive vector, or equivalently $(I - b)^{-1}$ exists and is non-negative, or $\lim_{k\to\infty} b^k = 0$. If every entity has non-zero final demand, all row sum to less than one, spectral radius is $\sigma(b) < 1$ and we get convergence. Otherwise this has to be checked in practice.

C.3 Environmentally-extended models

While C-Leontief and C-Ghosh model economic relationships, their environmentally-extended variants (EE-Leontief and EE-Ghosh) have become the dominant framework for environmental applications. These models serve two (similar) purposes: modeling EA use responses to exogenous variable changes (final demand for EE-Leontief; value-added for EE-Ghosh), and imputing EA use to specific components through component-wise calculations.

As established in Section 2, IH models offer distinct advantages over EE models for fine-scale EA footprinting: they compute imputations in a single operation without decomposition ; and they assign EA responsibility across all economic transactions, not just terminal variables. We nevertheless include these EE models in our appendix given their entrenched position in both literature and practical applications, serving as an important benchmark for comparison.

EA Intensity Factors in EE Models. EE models treat EA use as an *externality* of economic activity rather than as formal inputs/outputs in an IOA table formulated in EA use units. The models use fixed *direct* EA intensity factors represented either as row vector γ_J (size $1 \times |\mathbb{E}|$), or as column vector $\gamma_I = \gamma_J^\top \stackrel{\text{def}}{=} \gamma$ (size $|\mathbb{E}| \times 1$), with units of EA per monetary production. These relate to EA use through Hadamard products:

$$\phi_{d_J} = \gamma_J \odot \chi_J \tag{61}$$

$$\phi_{d_I} = \gamma_I \odot \chi_I \tag{62}$$

where ϕ_{d_J} and ϕ_{d_I} represent the same EA use vector in row and column forms respectively. For simplicity we use implicit notation and note ϕ_d and γ when form is clear from the context.

C.3.1 EE-Leontief model

EE-Leontief data:

$$\tilde{X}_{IJ} = \begin{array}{cc} \text{i} \setminus \text{j} & \begin{array}{cc} |\mathbb{E}| & \quad 1 \end{array} \\ \begin{array}{c} |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c|c} \chi & \chi_f \\ \hline \chi_w & 0 \end{array}\right) \end{array} \xrightarrow[\sum_j]{} \begin{pmatrix} \chi_I \\ \chi_d^{tot} \end{pmatrix} \tag{63}$$

$$\Big\downarrow \textstyle\sum_i$$

$$\left(\chi_I^\top \;\middle|\; \chi_d^{tot} \right)$$

$$\phi_d = \gamma \odot \chi_I \tag{64}$$

Consistent with the *fixed technology hypothesis* of all Leontief models, the EE-Leontief model assumes that every production entity j keeps using the same proportion of inputs i - including input from the environment ϕ_{d_i} - when its production $\chi_{J_j} = \chi_{I_j}$ changes, translating into equation 62.

Replacing χ_I with its calculated value as a function of final demand from equation 52, we get the hypothetical EA use value if final demand χ_f were to change:

$$\phi_d = \gamma \odot (I - a)^{-1} \chi_f \tag{65}$$

Imputing EA use to *components* of final demand. To analyze specific contributions, we can decompose χ_f into n_c additive components:

$$\chi_f = \sum_{k=1}^{n_c} \chi_f^{(k)}$$

Applying the model separately to each component yields detailed, *per-EA-user* (over i) EA footprints for each component, effectively allocating responsibility across final demand components:

$$\phi_d^{(k)} = \gamma \odot (I - a)^{-1} \chi_f^{(k)} \tag{66}$$

Classically, this kind of decomposition is used to calculate *consumption-based* emissions [24, 23], where final demand splits into domestic consumption and exports ($\chi_f = \chi_f^{(d)} + \chi_f^{(e)}$). Alternatively, final demand can be decomposed into $n_c = n$ coordinate projections:

$$\chi_f^{(i)} = \begin{pmatrix} (0) \\ \chi_{f_i} \\ (0) \end{pmatrix}$$

Here, each $\phi_d^{(i)}$ represents the *product-specific* footprint - detailed per EA user. Aggregating these gives the *total* final demand footprint of all products:

$$\psi_f^{EE} = \begin{pmatrix} \sum_i \phi_d^{(1)} \\ \cdots \\ \sum_i \phi_d^{(n)} \end{pmatrix}$$

C.3.2 EE-Ghosh model

EE-Ghosh data:

$$\tilde{X}_{IJ} = \begin{array}{cc} \text{i} \setminus \text{j} & \begin{array}{cc} |\mathbb{E}| & \quad 1 \end{array} \\ \begin{array}{c} |\mathbb{E}| \\ 1 \end{array} & \left(\begin{array}{c|c} \chi & \chi_f \\ \hline \chi_w & 0 \end{array}\right) \end{array} \xrightarrow[\sum_j]{} \begin{pmatrix} \chi_I \\ \chi_d^{tot} \end{pmatrix} \tag{67}$$

$$\Big\downarrow \textstyle\sum_i$$

$$\left(\chi_I^\top \;\middle|\; \chi_d^{tot} \right)$$

$$\phi_d = \gamma \odot \chi_I^\top \tag{68}$$

Consistent with the *fixed sales structure* of all Ghosh models, the EE-Ghosh model assumes that every production entity i keeps producing the same proportion of outputs j - including output to the environment ϕ_{d_j} - when its production $\chi_{I_i} = \chi_{J_i}$ changes, translating into equation 61.

Replacing $\chi_J = \chi_I^\top$ with its calculated value as a function of valued added from equation 57, we get the hypothetical EA use if value added were to change:

$$\phi_d = \gamma \odot \chi_w (I - b)^{-1} \tag{69}$$

Imputing EA use to *components* of final demand. To analyse specific contributions, we can also decompose χ_w into n_c additive components:

$$\chi_w = \sum_{k=1}^{n_c} \chi_w^{(k)}$$

Again, applying the model separately to each component yields detailed, *per-EA-user* (over j) EA footprints for each component, effectively allocating responsibility across value added components:

$$\phi_d^{(k)} = \gamma \odot \chi_w^{(k)} (I - b)^{-1} \tag{70}$$

If the decomposition consists of coordinate projections:

$$\chi_w^{(j)} = \begin{pmatrix} -0- & \chi_{w_j} & -0- \end{pmatrix}$$

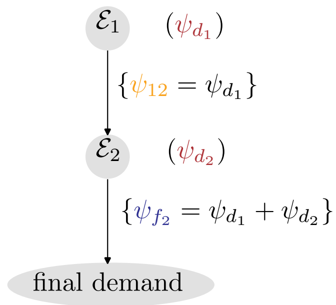
then each $\phi_d^{(j)}$ represents the *sector-specific* value added footprint - detailed per EA user. Aggregating these gives the *total* value added footprint of all sectors:

$$\psi_w^{EE} = \begin{pmatrix} \sum_j \phi_d^{(1)} & \dots & \sum_j \phi_d^{n} \end{pmatrix}$$

C.4 Simple examples for the IH-Ghosh model

To understand pollutant allocation, it can be useful to examine some simple examples. Figure 9 shows the most simple ‘descending’ economy with two groups, no within-sector sales, and final demand only for the downstream group. In this ‘fully descending’ case, pollution allocation is obvious: group $\mathcal{E}_2$ inheritates the footprint from upstream group $\mathcal{E}_1$ and passes it on to final demand d_2, adding its own direct emissions. In case of a similar descending structure with more groups, the final demand footprint is simply the sum of direct EA use of upstream entities. Figure 10 shows another limit case of a fully circular economy (here also with two sector and no within-sector sales) with no depth, i.e. every group provides final products. In this case, final product footprints are calculated from the IO table and direct EA use using equations 7 and 53.

(a) Entities $\mathcal{E}_i$ and transactions (arrows), with associated demand-side EA liability values ψ.

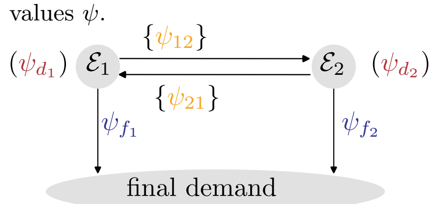


(b) Demand side EA imputation table (EA units)

	$\mathcal{E}_1$	$\mathcal{E}_2$	final
$\mathcal{E}_1$	0	$\psi_{12} = \psi_{d_1}$	0
$\mathcal{E}_2$	0	0	$\psi_{f_2} = \psi_{d_1} + \psi_{d_2}$
direct EA use	ψ_{d_1}	ψ_{d_2}	

Figure 9: Simple example of a fully descending economy with no within-group sales and final demand only for most downstream group

(a) Entities $\mathcal{E}_i$ and transactions (arrows), with associated demand-side EA liability values ψ.



(b) Demand-side EA use table (EA use units)

	$\mathcal{E}_1$	$\mathcal{E}_2$	final
$\mathcal{E}_1$	0	ψ_{12}	ψ_{f_1}
$\mathcal{E}_2$	ψ_{21}	0	ψ_{f_2}
direct EA use	ψ_{d_1}	ψ_{d_2}	

Figure 10: Simple example of a circular economy with no within-group sales, and final demand for both groups

D The Environmental Ledger (*e-ledger*)

In contrast to input-output-based EA imputation techniques - which perform *a posteriori* distribution using *global* transaction data - e-ledger approaches impute EA use *dynamically* during each financial transaction. This method relies exclusively on *local* information exchange between transacting parties, embedding EA accountability directly into transactional workflows.

Disclaimer On the term 'e-ledger': In this note, I use 'e-ledger' to refer to the general concept of setting environmental liability at the time of transaction. This is a broad methodological definition. It is not entirely clear to me how other initiatives—such as the E-Ledger Institute (which initially used a different name [74]) — precisely perform their calculations, and my use of the term may not match their specific technical meaning. A more precise term may be needed in the future to avoid confusion.

D.1 Fundamentals: e-ledger for demand-side EA responsibility imputation

As a *transactional allocation method*, the e-ledger operates at the granularity of *individual economic entities and products*. Like IO techniques, it presupposes monitored direct EA use by entities. The mechanism proceeds as follows: Any entity extracting EA resources receives a corresponding *EA use liability* on the environmental ledger Then, for each transaction, supplier and client entities negotiate the good/service exchanged, the monetary price, and the EA liability transfer amount. At transaction execution, the client receives liability for the agreed EA use amount, and the supplier's ledger balance is reduced correspondingly. The e-ledger enforces *conservation of liability units*: no liability may be created or destroyed, only transferred between entities.

More formally, consider entities i and j performing transaction $T_{ij,t}$ at time t, where j purchases a good/service from i, and j receives EA use liability $\psi_{ij,t} = \Delta\mathcal{P}_{ij,t}$. Let $\mathcal{P}_i(t)$ denote the liability balance of entity i at time t, with $t-$ and t^+ representing instants immediately before and after t (such that no intervening transactions occur). The e-ledger enforces:

$$\Delta\mathcal{P}_{ij,t} = \mathcal{P}_i(t^-) - \mathcal{P}_i(t^+) = \mathcal{P}_j(t^+) - \mathcal{P}_j(t^-) \tag{71}$$

In the e-ledger framework, EA use liabilities transferred to final-demand consumers become *de facto* product footprints. This system operates much like an 'additional currency' tied to environmental pressure. Like money, the e-ledger requires that units cannot be created or destroyed by participants, and depends on trusted institutional oversight—banking systems for money (where currency is created through debt without physical constraints), specialized platforms for EA liabilities. Just as with monetary systems, the entire structure collapses if participants lose trust in the ledger's integrity. The key difference is that while money exists as an abstract representation of value created through financial mechanisms, e-ledger units remain physically tied to real environmental resource use.

These structural parallels between e-ledgers and monetary systems likely explain why accounting professionals have been prominent proponents of this approach. Leading theoretical proposals include the E-Liability Institute's framework[74, 67] and Institut Messine's 'generalized carbon accounting' concept[84]. The core idea has also gained traction across different communities under various names[72, 89], reflecting growing academic and professional interest. While still in early development, private carbon accounting services are now exploring pilot applications of e-ledger principles[88, 86], suggesting potential pathways for future implementation.

In a *fully-downstream* economy—that is, one with *no loops* in production chains—the e-ledger concept emerges naturally, explaining its independent formulation by multiple researchers. Under these conditions, if each entity uniformly passes all received EA liability units to its clients, the system achieves inherent consistency: entities remain neutral (retaining no permanent liability), product footprints automatically reflect cumulative EA use with perfect accuracy, and their sum precisely equals total EA use.

D.2 Properties and remarks

The e-ledger approach in this basic form exhibits two defining characteristics. First, it operates exclusively at the *entity scale* - the finest granularity possible for legally distinct organizations, as inter-entity accounting systems would be both impractical and legally untenable. Second, and more fundamentally, it is inherently *local*: EA liability allocation requires only bilateral agreement between transacting parties, with no need for global coordination. These properties yield five important implications:

1. **No balance by design**. The basic e-ledger framework imposes no *balance constraint* on liability flows: for example, in the simplest e-ledger version, entities may allocate more or fewer EA liability units to products than they receive from suppliers during any time segment, provided their clients consent to such imbalances. This means source-allocated liabilities may not match product-allocated liabilities within arbitrary accounting periods.

   Remark Even in more constrained implementations - such as those requiring entity neutrality over accounting periods or allocating e-liabilities according to entity process data - e-ledger approaches cannot achieve the inherent balance of IOA. The fundamental limitation is that e-ledgers cannot resolve economic *loops* except through multiple sequential accounting periods. This reflects a deeper methodological distinction: IOA performs iterative refinements *offline* within each time segment, allowing convergence to a balanced solution before reporting. In contrast, e-ledgers operate *online* with exactly one iteration per accounting period, forcing any balancing to occur across multiple periods through market interactions.

2. **Fewer ties to physical reality**. Consequently, the basic e-ledger provides no guarantee that EA use imputed to a product corresponds to the physical resource requirements calculated via input-output (IO) analysis.

3. **EA Liability Units as Specialized Currency**. EA liability units function as a specialized currency where monitored EA extraction governs their creation, parties freely negotiate and exchange them, and entities actively accumulate or decumulate units in their ledger accounts. Like financial treasurers, managers must strategically balance these liabilities—unchecked accumulation may trigger insolvency when liability reserves exceed an entity's capacity to offset them through transactions or EA credits.

4. **Emergent EA Price Discovery**. When applying monetary-like constraints to e-ledgers (e.g., 'Borrowing limits' or 'Liability settlement requirements'), entities initially lack reference points for appropriate EA pricing in transactions. However, through iterative market interactions—where oversupplied liabilities depress prices and undersupplied ones increase them - the hope is that the system would experimentally converge to equilibrium 'market EA prices'. At these prices, most entities naturally achieve long-term liability neutrality, balancing received and allocated units over extended operational periods.

5. **Supply-Side E-Ledgers: An Unexplored Variant**. To our knowledge, all published work on the e-ledger adopt the demand-side framing presented here - where entities accumulate *expenditure-side e-liabilities* (our term) from suppliers - alongside direct EA use - and propagate these downstream to clients. However, we argue that the system could alternatively operate in supply-side mode through *income-side e-liabilities* - where entities preemptively distribute e-liabilities upstream to suppliers when purchasing inputs, before receiving client payments and e-liabilities. This inverse approach remains absent from both literature and practice, likely due to its counterintuitive mechanics: while demand-side allocation assigns e-liabilities to products *after* EA use occurs (through manufacturing inputs), supply-side e-ledgers would require entities to anticipate and allocate liabilities to suppliers *before* receiving them from clients. While this fundamental temporal reversal introduces operational challenges in forecasting and liquidity management, there is no fundamental reason why it could not function symmetrically to the demand-side version.

The vanilla e-ledger's simplicity manifests in two key aspects: First, beyond the basic conservation law of Equation 71, it requires virtually no additional mathematical formalism[19], and second, it operates continuously without requiring predefined *time periods* for EA balancing. This temporal freedom eliminates the accounting constraints inherent to IOA's discrete-time structure.

D.3 Comparing e-ledger and IOA

Complementary strengths. The environmental ledger (e-ledger) and input-output analysis (IOA) offer fundamentally different approaches to environmental accounting, with complementary strengths summarized in Table 2. The e-ledger operates through continuous, transaction-level accounting. Entities exchange EA liabilities in real time, building or drawing down their environmental balances. This approach requires only bilateral agreements and ledger integrity, but provides no equilibrium guarantees—entities may persistently accumulate or decumulate liabilities rather than reaching balance. IOA takes the opposite approach, ensuring strict equilibrium by design. Its periodic calculations perfectly allocate all environmental impacts across the economy—but only retrospectively. Participants must rely on historical data until accounting periods close, lacking real-time EA values during transactions. In summary, the e-ledger enables dynamic, decentralized accounting, while IOA delivers complete equilibrium at the cost of temporal resolution.

allocation method	**e-ledger**	**IOA**
when is EA amount ψ_{ij} determined ?	before transaction	'a posteriori' after a given time period
who does ψ_{ij} depend on ?	only entities i and j	all entities in the economy
how is ψ_{ij} determined ?	by 'agreement' between i and j, similar to price value [a]	by the model
equilibrium/physical sense guarantees ?	no	yes, by design
how are 'sensible' EA values obtained ?	by 'trial and error', converges to 'market values'	by design of the model
data collection granularity required ?	fine-scale (entity-to-entity transactions)	fine-scale or group-scale
demand-side and supply-side imputation both possible ?	yes, but supply-side is less natural	yes

Table 2: Comparison between the e-ledger and the IOA approaches for computing EA consumption footprints.

[a] This is for the *vanilla* version of the e-ledger. More constrained implementations may limit the set of allowed allocations - but essentially remain an *a-priori* agreement, as opposed to an *a-posteriori* calculation.

Synergistic potential Rather than competing, these methods complement each other naturally. IOA's global scope can enforce cross-scale consistency, guarantees equilibrium (ensuring no EA use remains unallocated), and provides initial EA benchmarks for e-ledgers. Conversely, e-ledgers generate certified, market-validated data that can support policy implementation.

Deploy now, upgrade later. For practical deployment, we advocate beginning with fine-scale IOA extensions for two key reasons. First, IOA's theoretical maturity and existing implementations require 'only' granular data integration - a significant but tractable extension compared to building an entirely new e-ledger infrastructure. Second, the system offers flexible evolution: continuous-time IOA developments may inherently replicate e-ledger advantages, or e-ledgers could later be incorporated strategically where their real-time verification provides demonstrable benefits. The metacrisis leaves no

[19] This contrasts with IOA's extensive global consistency framework—a need the e-ledger bypasses by design.

time for paralysis - we must act now with IOA's ready toolkit, *build the first generation of accountable systems within a few years*, and iteratively upgrade with e-ledger-type capabilities as implementations and methods mature.

E GHG Protocol

The GHG Protocol (GHGP) provides the world's most widely adopted greenhouse gas accounting standards, having established corporate emissions monitoring and spurred the development of global emission factor databases. While acknowledging its foundational role - particularly the rigor of its Scope 1 framework - we demonstrate why the GHGP approach cannot serve as the basis for a comprehensive EA footprinting system. This analysis should not be misconstrued as criticism: these methods represented the optimal solution given the data availability and methodological developments of their time. However, as environmental accountability requirements evolve, we argue that today's challenges demand advanced footprinting methods that build upon, rather than replicate, the GHGP's pioneering framework. Our approach builds on the Protocol's strengths while overcoming its limitations for tomorrow's EA accounting.

E.1 Fundamentals

The GHG Protocol [83] establishes standardized methods for quantifying and reporting greenhouse gas emissions. Its core *Corporate Standard* operates at the entity level, requiring companies to assess only their own purchases within defined organizational boundaries. The methodology follows three sequential steps: (1) creating an inventory of all entity purchases, (2) categorizing each purchased item, and (3) calculating embedded footprints using category-specific emission factors [20] The Standard provides comprehensive guidance on boundary-setting, emission factor sourcing, and the classification of emissions into Scopes 1-3. This framework has been adopted as the basis for national standards in many countries.

The critical feature of GHG inventories is their *entity-level perspective*. The resulting aggregate value represents what we term an *involvement value* (see Section 2.4), capturing all Environmental Assets (here, greenhouse gases) that interact with the entity's operations. This approach connects measurement to action - by focusing on purchased inputs, it identifies precisely those flows the entity can control, while maintaining accountability through clear inventory boundaries. While this approach provides a workable foundation for emissions accounting, later sections will show how more sophisticated methods can better capture the full complexity of environmental impacts.

The GHG Protocol has expanded to include additional standards addressing various accounting scenarios. These encompass specialized guidance for Scope 3 emissions estimation and frameworks for assessing non-corporate entities such as cities [80], policies [85], and products [87]. All these standards maintain the same fundamental approach: (1) establishing appropriate boundaries for the subject of study, and (2) applying emission factors to input inventories to calculate involvement values. This methodological consistency allows for comparable results across different applications while adapting to specific accounting needs.

E.2 Emission factors

The ecosystem of emission factors (EFs) remains to be addressed. Entities typically source EFs from several authoritative channels. The IPCC database [76] offers conservative, sector-level default values when no alternatives exist. National inventories [36, 70, 65] provide country-specific factors accounting for regional conditions like energy mixes and industrial practices. For greater precision, sector-specific databases [46, 75, 38] deliver process-tailored factors developed by industry groups. When dealing with unique, undocumented processes, entities are advised to conduct direct emissions measurements rather than relying on generalized factors.

The calculation methodologies for emission factors vary significantly by emission category. Direct emissions typically permit precise measurement or estimation: methane emissions from landfills can be quantified through on-site gas capture, nitrous oxide from agricultural soils via field measurements, pipeline emissions through input-output differentials, and cement emissions via stoichiometric analysis of raw materials. Combustion-related factors derive directly from reaction chemistry. For indirect

[20] While guidelines include *Scope 3 downstream* emissions (e.g., distribution, use, and disposal of sold products), these are negligible for our EA footprinting focus and thus omitted here.

emissions, such direct measurement proves impossible. Scope 2 electricity factors combine generation source emissions weighted by their grid contribution, adjusted for regional power transfers. Scope 3 emissions (including purchased goods, capital equipment, and upstream logistics) require input-output modeling across economic sectors.

E.3 Remarks with regards to EAs footprinting

Having outlined the core principles of GHG Protocol standards, we now contextualize these methods relative to our central objective: establishing robust Environmental Asset footprinting for final demand products and value added revenue. Five key considerations emerge from this comparison.

1. **Consistency and Double Counting** The decentralized development of GHG Protocol emission factors (EFs) creates inherent inconsistencies across entities, categories, and scales. Independent EF creation across different databases, scopes, and sectors forces unavoidable aggregation discrepancies. The gasoline supply chain illustrates this perfectly: Country A and Country B each calculate separate upstream EFs while sharing extraction processes in Country C, with no built-in mechanism to align their methodologies. Ad hoc solutions like purchase-volume prorating can bridge some gaps, but these rely on voluntary cooperation rather than the *consistent by design* framework needed for robust accounting. Complex global supply chains magnify these issues, as temporal variations and sectoral methodology differences compound across national boundaries. Only simultaneous EF development within an integrated allocation framework - with explicit application rules - can deliver true consistency. This is precisely what Impact-Inheritance Input-Output models achieve through their comprehensive accounting structure.

2. **Static Character** The inherent development process of emission factors (EFs) imposes fundamental limitations on their dynamism. Developing each EF demands prohibitively expensive case studies, forcing rare database updates that struggle to reflect our rapidly evolving economic landscape. Geopolitical realignments, technological breakthroughs, natural events, and competitive pressure constantly rewrite production networks, yet EFs remain frozen in time. The protocol compounds this rigidity by shackling emissions allocation to rigid categorical boxes. We need multidimensional frameworks that simultaneously support alternative EF sets for diverse economic analyses, adapt to new classification schemes as they emerge, and continuously incorporate fresh data - capabilities fundamentally absent from current standards.

3. **Excessive Averaging and Non-Specificity** Scope 1 emission factors maintain reasonable accuracy, but indirect emission factors suffer from inherent estimation challenges. Current methodologies lack the tools to trace emissions through company-to-product pathways, while data limitations force Environmentally-Extended Input-Output (EEIO) models to rely on heavily aggregated sectors. These constraints systematically prevent development of precise emission factors for specific products or narrow sectors, particularly for Scope 3 emissions. Consequently, available factors represent broad averages that obscure potentially important variations across product features and sub-sectors.

4. **Lack of Actionability** These systemic flaws - inconsistency, temporal rigidity, and excessive aggregation—render GHG Protocol outputs unfit for regulatory implementation. The methodology invites manipulation through EF selection bias, where actors can strategically choose emission factors that minimize reported footprints. This fundamental lack of enforceability represents the standards' critical weakness: no rational actor would accept binding constraints based on such malleable metrics. Effective constraint systems require allocation frameworks with rigorous mathematical properties - source certification, flux conservation, and unambiguous allocation rules - none of which the current approach guarantees. The protocol's outputs remain useful for voluntary reporting but fail the robustness test for policy implementation.

5. **Challenges in estimating the impact of value-added activities**. As outlined earlier, an individual's environmental impact can be assessed from two complementary perspectives: (1) the impact generated through the purchase of final-demand products, and (2) the impact driven by value-added activities (e.g., labor, investments). Both approaches are necessary, as they represent two sides of the same reality: product purchases create demand-side emissions,

while value-added activities enable supply-side emissions. However, the GHG Protocol standards were never designed to support *upstream impact attribution*—estimating emissions tied to value-added contributions like wages or investment returns. Instead, they focus on organizational- and product-level accounting, leaving a gap in linking emissions to income streams. This represents a key methodological limitation.

6. **Source Definition Strategy** Section 1.3 established that all footprinting methods must first define source processes - physical operations that either directly emit pollutants or create products with quantifiable emission potential. The GHG Protocol operates through product categories, strategically choosing which upstream processes to bundle into Scope 1 emission factors. Each factor represents an aggregate over all production chains for that category (or those sampled in EF studies), regardless of their individual environmental variations. For example, the Protocol's 'gasoline' category bundles all production pathways into one emission factor. This lets a trucking company report both tailpipe emissions and averaged upstream impacts under Scope 1, despite two key limitations: (1) physical misattribution, since the company only controls combustion, not extraction or refining; and (2) aggregation error, as the company influences just its suppliers' practices, not the industry-wide average. A more precise approach would define sources at extraction, making only tailpipe emissions Scope 1—but at the cost of reduced apparent control.

   The Protocol's framework - complemented by national emission registries under UNFCCC guidelines - provides the *essential foundation for EA accounting.* These systems offer precisely measured direct emissions data, creating the necessary basis for implementing the EA footprinting system developed in this work. As implementation progresses, source definitions can move systematically upstream—shifting from category averages to supplier-specific allocations where supported by emerging data streams.

In summary, while GHG Protocol inventories could theoretically allocate entity-level emissions to product footprints, the results would be unreliable. More critically, the framework inherently lacks the capability to impute emissions to value-added revenue streams. Without traceability across economic transactions, the Protocol defaults to coarse and rigid sector averages - an inadequate basis for effective emission attribution. Truly actionable EA footprints demand a purpose-built system featuring: (1) granular source tracking, (2) dynamic allocation rules, and (3) entity-level consistency controls - precisely the architecture our method delivers.

F The metacrisis and the great simplification

The interconnected web of systemic global crises - climate collapse, ecological degradation, geopolitical instability, technological disruption, economic inequality, and value system failures - is sometimes called *metacrisis.* Components of it cause and reinforce each other.

The great simplification Unless the mindless superorganism rapidly transforms into a frugal, mindful, and collaborative society - a shift requiring unprecedented social advancement - our current systems will collapse within a human-relevant timescale: the *great simplification* (term by Hagens), a forced return to simpler, lower-energy societal structures. Already, daily signs - from climate-driven disasters to supply chain shocks and escalating conflicts - suggest our global economy has entered this metacrisis phase [90, 63, 43]. The good news is that beyond basic needs, EA consumption becomes decoupled from human well-being - revealing significant potential for reduction without compromising quality of life [44].

Unknown dynamics The trajectory of the metacrisis and *great simplification* remains highly uncertain due to two nonlinear phenomena. First, environmental systems exhibit tipping points [21, 28, 17], where minor incremental changes can trigger abrupt, irreversible collapse. Second, our financial system's instability - built on debt as a claim on *future energy expenditure* [43, 55, 32] - is prone to a *Minsky moment* [10]. Such a collapse could occur decades before biophysical limits are reached, as debt defaults expose the illusion of perpetual growth in a resource-constrained world.

Our current response Our current response to the emerging metacrisis remains dangerously 'mindless', constrained by cognitive biases [59], short-term debt dynamics [40], and the net-zero illusion: a *short-boundary lens* [22] that ignores both the statistical near-impossibility of decarbonizing without systemic economic change [43, 37], and the accelerating environmental shifts that will disrupt even 'optimal' transition pathways [56]. *'We must change economic systems first'* [43, 49] - yet policy clings to fantasies.

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